States and Observables in Hamiltonian Semiclassical Scalar Electrodynamics

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hep-th/0412323

Abstract

The main notions of semiclassical scalar electrodynamics in different gauges (Hamiltonian, Couloumb, Lorentz) are discussed. These are semiclassical states, Poincare transformations, fields, observables, gauge equivalence. General properties of these objects are formulated as axioms of semiclassical theory; they are heuristically justified. In particular, a semiclassical state may be viewed as a set of classical background field and quantum state in the external background. Superpositions of these "elementary" states can be also considered. Set of all "elementary" semiclassical states forms a semiclassical bundle, with base being classical space and fibres being quantum states in the external background. Quantum symetry transformations (Poincare and gauge transformations) are viewed semiclassically as automorphisms of the semiclassical bundle. Specific features of electrodynamics are investigated for different gauges.

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[0]This work was supported by the Russian Foundation for Basic Research, project 02-01-01062

1 Introdcution

States and observables (fields) are main notions of the axiomatic quantum field theory (QFT). The difficulty is that it is unknown whether a nontrivial model of axiomatic AFT exists in 4 dimensions [1]. All practical calculations in QFT (evaluations of scattering and decay properties) are performed using the heuristic Lagrangian framework. All the axioms of QFT are checked within the perturbation theory [1, 2, 3, 4]: if a formal perturbation series for physical quantities satsfies the set of axioms then one says that axioms are checked.

Another important heuristic approximate method is a semiclassical approximation. A lot of examples of physical applicatons of the semiclassical approximation are known: these are soliton quantization theory [5, 6], QFT in a strong external background classical field [7] or in curved space-time [8], the one-loop approximation [9], time-dependent Hartree-Fock [9, 10] and Gaussian approximations [11].

However, the main axiomatic notions (states, observables and fields), as well as correspondence principle between quantum and classical field theories are to be clarified. For the scalar field theories, the axioms of semiclassical field theory were suggested in [12]. The purpose of this paper is to formulate and investigate analogs of these axioms for gauge theories. Quantum electrodynamics (QED) is considered as an example of an Abelian gauge theory. Since one knows formulations of QED in different gauges (Hamiltonian, Couloumb, Lorentz), the corresponding formulations of the semiclassical theory should be investigated. One expects all the formulations to be equivalent; these equivalence should be checked then.

Section 2 deals with properties of states and observables in semiclassical field theory. The discussion is based mostly on refs. [12, 13]. In section 3 different approaches of quantizing electrodynamics are reviewed. Section 4 is devoted to the notion of semiclassical state for different gauges. In section 5, semiclassical observables and transformations are investigated. Section 6 contains concluding remarks.

2 Properties of states and observables in semiclassical field theory

Let us discuss general properties of semiclassical field theory. Consider a simpler example of scalar field theory with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{h}V(\sqrt{h}\varphi), \tag{2.1}$$

h being a small parameter of expansion.

1. A "naive" semiclassical theory can be constructed as follows (cf. [7]). One extracts a c-number component $\Phi(x)/\sqrt{h}$ from the field $\varphi(x)$:

$$\varphi(x) = \frac{\Phi(x)}{\sqrt{h}} + \phi(x); \tag{2.2}$$

then the remaining part $\phi(x)$ is quantized. Substitution (2.2) to the Lagrangian (2.1) leads to the following action

$$\begin{gathered} I = \frac{1}{h}\int dx \left[\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - V(\Phi)\right] + \frac{1}{\sqrt{h}}\int dx \left[\frac{1}{2}\partial_\mu\Phi\partial^\mu\phi - V'(\Phi)\phi\right] \\ + \int dx \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}V''(\Phi)\phi^2\right] + ... \end{gathered}$$

The term of the order $O(1/h)$ is constant and can be omitted; the second term (linear in ϕ) vanishes due to classical equation of motion for Φ; the remaining quadratic term is

$$I_2 = \int dx \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}V''(\Phi)\phi^2\right]. \tag{2.3}$$

Then action (2.3) is quantized, and a semiclassical theory is obtained.

2. The semiclassical theory can be also constructed in the Hamiltonian approach as well. One considers the quantum theory correspoinding to the Lagrangian (2.1) and investigates the semiclassical states which depends on the small parameter h due to the Maslov substitution (its analog was suggested for quantum mechanical problems in [14, 15]):

$$\Psi \simeq e^{\frac{i}{h}\tilde{S}} e^{\frac{i}{\sqrt{h}}\int d\mathbf{x}[\Pi(\mathbf{x})\hat{\varphi}(\mathbf{x}) - \Phi(\mathbf{x})\hat{\pi}(\mathbf{x})]} f \equiv \tilde{K}^h_{\tilde{S},\Pi,\Phi} f. \tag{2.4}$$

Here $\hat{\pi}(\mathbf{x})$ is a momentum canonically conjugated to the field $\hat{\varphi}(\mathbf{x})$. In the functional Schrodinger representation (the field and momentum operators are $\hat{\varphi}(\mathbf{x}) = \varphi(\mathbf{x})$, $\hat{\pi}(\mathbf{x}) = -i\frac{\delta}{\delta\varphi(\mathbf{x})}$, states are $\Psi[\varphi(\cdot)]$) formula (2.4) can be rewritten as

$$\Psi[\varphi(\cdot)] = const e^{\frac{i}{h}S} e^{\frac{i}{\sqrt{h}}\int d\mathbf{x}\Pi(\mathbf{x})[\varphi(\mathbf{x})\sqrt{h} - \Phi(\mathbf{x})]} f[\varphi(\cdot) - \frac{\Phi(\cdot)}{\sqrt{h}}] \equiv K^h_{S,\Pi,\Phi} f[\varphi(\cdot)] \tag{2.5}$$

with $S = \tilde{S} + \frac{1}{2}\int d\mathbf{x}\Pi\Phi$. If theclassical field is $\varphi(\mathbf{x}) = \Phi(\mathbf{x})/\sqrt{h} + O(1)$ then the probability amplitude (2.5) is large; otherwise, for the case $\varphi - \Phi/\sqrt{h} = O(1/\sqrt{h})$, it is exponentially small. Therefore, $\Phi(\mathbf{x})/\sqrt{h}$ may be viewed as a classical component of the field.

The set of semiclassical states (2.5) may be treated as *a bundle* ("semiclassical bundle" [16]), the base of the bundle is $\{X \equiv (S, \Pi(\cdot), \Phi(\cdot))\}$ - a set of classical states; the fibres

$\mathcal{F}_X = \{f\}$ are state spaces in given external fields X. The operator $K_X^h : f \mapsto \Psi$ is called as *a canonical operator.*

3. A specific feature of the Hamiltonian approach to semiclassical field theory is that ine can investigate also states of the more general form than (2.5). Namely, one can consider superpositions of the form (cf.[17])

$$\int d\alpha K^h_{X(\alpha)} f(\alpha), \qquad \alpha = (\alpha_1, ..., \alpha_k), \tag{2.6}$$

which can be viewed as k-dimensional surfaces of the semiclassical bundle. Such superpositions are useful in the soliton quantization theory due to the well-known problem of zero modes of the solitons. In quantum mechanics, one can obtain [17] the WKB method and all Maslov methods of [14, 15] from the wave packet method with the help of using superpositon (2.6).

One can call the state $K_X^h f$ as an "elementary semiclassical state", while superposition (2.6) can be interpreted as a "composed semiclassical state".

It is necessary to investigate the following problems within the semiclassical theory:

- action of Poincare transformations $\mathcal{U}_g^h$ (in particular, evolution) corresponding to elements g of the Poincare group G;

- action of Heisenberg field operators $\hat{\varphi}(x)$;

- inner product of states (2.6).

4. It happens that the following commutation rules are satisfied as $h \to 0$:

$$\begin{gathered} \mathcal{U}_g^h K_X^h f \simeq K^h_{u_g X} U_g(u_g X \leftarrow X) f; \\ \sqrt{h}\hat{\varphi}(x) K_X^h f \simeq K_X^h [\Phi(x|X) + \sqrt{h}\hat{\phi}(x|X)] f. \end{gathered} \tag{2.7}$$

Here $u_g : X \mapsto u_g X$ is a classical Poincare transformation, $\Phi(x|X)$ is a classical field corresponding to the classical state X. $\Phi + \sqrt{h}\hat{\phi}$ may be viewed as a semiclassical field.

An explicit form of the semiclassical Poincare transformation $U_g(u_g X \leftarrow X)$ was constructed in [12].

Important properties of classical and semiclassical fields for the model (2.1) may be obtained from the Heisenberg equations

$$\partial_\mu \partial^\mu \sqrt{h}\hat{\varphi}(x) + V'(\sqrt{h}\hat{\varphi}(x)) = 0.$$

Making use of (2.7), one finds that

$$\begin{gathered} \partial_\mu \partial^\mu \Phi(x|X) + V'(\Phi(x|X)) = 0; \\ \partial_\mu \partial^\mu \hat{\phi}(x|X) + V''(\Phi(x|X))\hat{\phi}(x|X) = 0. \end{gathered} \tag{2.8}$$

Eqs. (2.8) should be completed by the initial conditions at $t=0$:

$$\Phi(\mathbf{x}|X)|_{t=0} = \Phi(\mathbf{x}); \quad \dot{\Phi}(\mathbf{x}|X)|_{t=0} = \Pi(\mathbf{x});$$
$$\hat{\phi}(\mathbf{x}|X)|_{t=0} f[\phi(\cdot)] = \phi(\mathbf{x}) f[\phi(\cdot)]; \qquad \dot{\hat{\phi}}(\mathbf{x}|X)|_{t=0} f[\phi(\cdot)] = \frac{1}{i}\frac{\delta}{\delta\phi(\mathbf{x})} f[\phi(\cdot)].$$

Investigate properties of Poincare transformations. Since operators $\mathcal{U}_g^h$ should satisfy the group identity

$$\mathcal{U}_{g_1 g_2}^h = \mathcal{U}_{g_1}^h \mathcal{U}_{g_2}^h,$$

it follows from relation (2.7) that

$$\begin{gathered} u_{g_1 g_2} = u_{g_1} u_{g_2}; \\ U_{g_1 g_2}(u_{g_1 g_2} X \leftarrow X) = U_{g_1}(u_{g_1 g_2} X \leftarrow u_{g_2} X) U_{g_2}(u_{g_2} X \leftarrow X). \end{gathered} \tag{2.9}$$

Properties (2.9) mean that the Poincare group acts on the semiclassical bundle as an automorphism group.

For the Poincare transformation $g = (a, \Lambda)$ of the form $x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu$, it follows from Poincare invariance of the fields that

$$\mathcal{U}_{g^{-1}}^h \hat{\varphi}(x) \mathcal{U}_g^h = \hat{\varphi}(w_g x), \qquad w_g x = \Lambda^{-1}(x-a). \tag{2.10}$$

therefore, one obtains Poincare invariance property of classical and semiclassical fiedls:

$$\begin{gathered} \Phi(x|u_g X) = \Phi(w_g x|X); \\ \hat{\phi}(x|u_g X) U_g(u_g X \leftarrow X) = U_g(u_g X \leftarrow X) \hat{\phi}(w_g x|X). \end{gathered} \tag{2.11}$$

5. Consider the inner product (Ψ, Ψ) for the composed semiclassical state Ψ (2.6). One can calculate it as follows [13]: write it as

$$(\Psi, \Psi) = \int d\alpha d\alpha' (K_{X(\alpha)}^h f(\alpha), K_{X(\alpha')}^h f(\alpha')), \tag{2.12}$$

consider the substitution $\alpha' = \alpha + \sqrt{h}\beta$, expand the expression in $\sqrt{h}$. However, it is necessary to write an expansion for the state $K_{X(\alpha+\beta\sqrt{h})}^h f(\alpha + \beta\sqrt{h})$ into a series in $\sqrt{h}$. It can be obtained from the commutation rule between operators $ih\frac{\partial}{\partial\alpha_a}$ and $K_{X(\alpha)}^h$:

$$ih\frac{\partial}{\partial\alpha_a} K_{X(\alpha)}^h f \simeq K_{X(\alpha)}^h \left[\omega_{X(\alpha)}[\frac{\partial X}{\partial\alpha_a}] - \sqrt{h}\Omega_{X(\alpha)}[\frac{\partial X}{\partial\alpha_a}] + ... \right] f. \tag{2.13}$$

The c-number 1-form $\omega_X[\delta X]$ ('action form') and the operator-valued 1-form $\Omega_X[\delta X]$ (acting in $\mathcal{F}_X$) are important objects of the semiclassical theory. Their explicit form is

$$\begin{gathered}\omega_X[\delta X] = \textstyle\int d\mathbf{x}\Pi(\mathbf{x})\delta\Phi(\mathbf{x}) - \delta S; \\ \Omega_X[\delta X]f[\phi(\cdot)] = \textstyle\int d\mathbf{x}[\Pi(\mathbf{x})\phi(\mathbf{x}) - \Phi(\mathbf{x})\frac{1}{i}\frac{\delta}{\delta\phi(\mathbf{x})}]f[\phi(\cdot)].\end{gathered} \tag{2.14}$$

It follows from the relation $[ih\frac{\partial}{\partial\alpha_a}; ih\frac{\partial}{\partial\alpha_b}] = 0$ that the commutator of operators Ω should be a c-number:

$$\left[\Omega_{X(\alpha)}[\frac{\partial X}{\partial\alpha_a}]; \Omega_{X(\alpha)}[\frac{\partial X}{\partial\alpha_b}]\right] = -i\left\{\frac{\partial}{\partial\alpha_a}\omega_{X(\alpha)}[\frac{\partial X}{\partial\alpha_b}] - \frac{\partial}{\partial\alpha_b}\omega_{X(\alpha)}[\frac{\partial X}{\partial\alpha_a}]\right\}. \tag{2.15}$$

One can also write relation (2.15) in a shorter form. Namely, the c-number commutator is related with the symplectic 2-form $d\omega$:

$$[\Omega_X[\delta X_1]; \Omega_X[\delta X_2]] = -id\omega_X(\delta X_1, \delta X_2). \tag{2.16}$$

Certainly, commutation relations (2.15) and (2.16) are satisfied for objects (2.14).

To find an explicit form of $K^h_{X(\alpha+\sqrt{h}\beta)}$, set

$$K^h_{X(\alpha+\sqrt{h}\beta)} = K^h_{X(\alpha)}V_h(\alpha,\beta).$$

One obtains the following equation on $V_h(\alpha,\beta)$:

$$\frac{\partial}{\partial\beta_a}V_h(\alpha,\beta) \simeq -\frac{i}{\sqrt{h}}V_h(\alpha,\beta)(\omega - \sqrt{h}\Omega)_{X(\alpha+\sqrt{h}\beta)}[\frac{\partial X}{\partial\alpha_a}(\alpha+\sqrt{h}\beta)]. \tag{2.17}$$

Therefore, in the leading order in h one obtains that the operator V_h is a multiplicator by a rapidly oscillating c-number

$$V_h(\alpha,\beta) \sim e^{-\frac{i}{\sqrt{h}}\beta_a\omega_{X(\alpha)}[\frac{\partial X}{\partial\alpha_a}]}.$$

The inner product (2.12) is taken to the form

$$h^{k/2}\int d\alpha d\beta(f(\alpha), V_h(\alpha,\beta)f(\alpha+\sqrt{h}\beta)). \tag{2.18}$$

The integrand in (2.18) rapidly oscillates, so that the integral is exponentially small, except for the special case

$$\omega_{X(\alpha)}[\frac{\partial X}{\partial\alpha_a}] = 0. \tag{2.19}$$

Therefore, one should consider not any superposition (2.6) but superpositions obeying *the Maslov isotropic condition (2.19).* Only for this case the composed semiclassical state is not exponentially small.

If condition (2.19) is satisfied, one can solve eq.(2.17), provided that the commutation relation (2.15) is obeyed. One has

$$V_h(\alpha,\beta) \sim e^{i\beta_a \Omega_{X(\alpha)}[\frac{\partial X}{\partial \alpha_a}]}.$$

Therefore, for (Ψ,Ψ) one has

$$(\Psi,\Psi) \simeq h^{k/2} \int d\alpha d\beta (f(\alpha), \prod_a \{2\pi\delta(\Omega_X[\frac{\partial X}{\partial \alpha_a}])\} f(\alpha)). \tag{2.20}$$

One can notice that it is necessary to multiply the composed state (2.6) by $h^{-k/4}$ in order to satisfy the normalization condition.

Combining commutation rules (2.7) and (2.13), one obtains new identities:

$$\begin{gathered} \omega_X[\tfrac{\partial X}{\partial \alpha_a}] = \omega_{u_g X}[\tfrac{\partial (u_g X)}{\partial \alpha_a}]; \\ U_g(u_g X \leftarrow X)\Omega_X[\tfrac{\partial X}{\partial \alpha_a}] = \Omega_{u_g X}[\tfrac{\partial (u_g X)}{\partial \alpha_a}] U_g(u_g X \leftarrow X); \end{gathered} \tag{2.21}$$

$$-i\frac{\partial}{\partial \alpha_a}\Phi(x|X) = [\hat{\phi}(x|X); \Omega_X[\frac{\partial X}{\partial \alpha_a}]]. \tag{2.22}$$

The first identity means that the classical symplectic structure should be invariant under Poincare transformations. The second equality is related with unitarity of Poincare transformations for the composed states (2.20).

6. Therefore, all the problems of semiclassical field theory in the leading order can be solved under the following conditions ("axioms of semiclassical theory"):

A1. *A semiclassical bundle is given; space of the bundle is interpreted as a set of semiclassical states; base $\mathcal{X} = \{X\}$ is a classical space; elements of fibres $\mathcal{F}_X$ are quantum states in a given classical external field X.*

A2. *The Poincare group G acts as an automorphism group on the semiclassical bundle; group properties (2.9) are satisfied.*

A3. *Classical and semiclassical components of the field $\Phi(x|X)$ and $\hat{\phi}(x|X)$ are given for all $X \in \mathcal{X}$. $\Phi(x|X)$ is a c-number classical field; $\hat{\phi}(x|X)$ is an operator distribution acting in $\mathcal{F}_X$. The property (2.11) of Poincare invariance of the field is satisfied.*

A4. *The differential 1-forms ω and Ω are given on $\mathcal{X}$; $\omega_X[\delta X]$ is a real c-number; $\Omega_X[\delta X]$ is an operator acting in $\mathcal{F}_X$. The commutation relation (2.15) for Ω and properties (2.21), (2.22) for 1-forms, fields and Poincare transformations are satisfied.*

Therefore, one can say that a model of semiclassical field theory is *given in the leading order* if the objects of axioms A1-A4 are specified and their properties are obeyed. For the semiclassical problems, *it is not important* whether the "exact" QFT model is well-defined mathematically or not.

7. The formulated axioms and properties are not independent. It happens that one can express the operator $\hat{\phi}(x|X)$ via the 1-form Ω.

Introduce manifestly covariant notations. Let us identify elements $X \in \mathcal{X}$ with sets $(S, \overline{\Phi}(x))$ (instead of $(S, \Pi(\mathbf{x}), \Phi(\mathbf{x}))$, where $\overline{\Phi}(x)$ is a solution of the Cauchy problem for classical field equaton

$$\partial_\mu \partial^\mu \overline{\Phi}(x) + V'(\overline{\Phi}(x)) = 0, \qquad \overline{\Phi}|_{t=0} = \Phi(\mathbf{x}), \quad \frac{\partial}{\partial t}\overline{\Phi}|_{t=0} = \Pi(\mathbf{x}).$$

Then elements δX of the tangent space should be identified with pairs $(\delta S, \delta\overline{\Phi}(x))$, with $\delta\overline{\Phi}(x)$ being a solution of variation equation

$$\partial_\mu \partial^\mu \delta\overline{\Phi}(x) + V''(\overline{\Phi}(x))\delta\overline{\Phi}(x) = 0. \tag{2.23}$$

Then

$$\omega[\delta X] = \int_{x^0=0} d\mathbf{x} \partial_0 \overline{\Phi}(\mathbf{x}) \delta\overline{\Phi}(\mathbf{x}) - \delta S.$$

Property (2.15) can be also taken to a manifestly covariant form:

$$\begin{array}{c} [\Omega_\Phi[\delta_1\overline{\Phi}]; \Omega_\Phi[\delta_2\overline{\Phi}]] = -i \int_{x^0=0} d\mathbf{x}[\delta_1 \partial_0 \overline{\Phi} \delta_2 \overline{\Phi} - \delta_1 \overline{\Phi} \delta_2 \partial_0 \overline{\Phi}] = \\ -i \int d\sigma^\mu [\partial_\mu \delta_1 \overline{\Phi} \delta_2 \overline{\Phi} - \delta_1 \overline{\Phi} \partial_\mu \delta_2 \overline{\Phi}]. \end{array} \tag{2.24}$$

Let us construct the operator $\hat{\phi}(y|X) \equiv \hat{\phi}(y|\overline{\Phi})$ from the relation (2.22):

$$-i\delta\overline{\Phi}(y) = [\hat{\phi}(y|\overline{\Phi}), \Omega_\Phi[\delta\overline{\Phi}]]. \tag{2.25}$$

One can notice from eq.(2.24) that the operator function

$$\hat{\phi}(y|\overline{\Phi}) = \Omega_\Phi[\delta\overline{\Phi}^{(y)}] \tag{2.26}$$

satisifes eq.(2.25), provided that $\delta\overline{\Phi}^{(y)}$ is a solution of eq.(2.23), such that the additional conditions of the form

$$\delta\overline{\Phi}^{(y)}_{x^0=y^0} = 0, \qquad \delta\partial_0\overline{\Phi}^{(y)}_{x^0=y^0} = \delta(\mathbf{x} - \mathbf{y}).$$

are satisfied. One can also express $\delta\overline{\Phi}^{(y)}(x)$ via the retarded Green function for eq.(2.23):

$$[\partial_\mu \partial^\mu + V''(\overline{\Phi}(x))] D_\Phi^{ret}(x,y) = \delta(x,y);$$
$$D_\Phi^{ret}(x,y) = 0, \quad x < y,$$

since

$$\delta\overline{\Phi}^{(y)}(x) = D_\Phi^{ret}(x,y), \quad x > y.$$

If the definition (2.26) is accepted, properties of Poincare invariance of fields are corollaries of properties of the operator Ω. The 1-forms seems then to be more important objects of the semiclassical theory than fields.

8. Let us discuss now general specific features of semiclassical gauge field theories. It happens that some of classical states may be *gauge-equivalent*: $X_1 \sim X_2$ [19]. This means that semiclassical states $K^h_{X_1} f_1$ and $K^h_{X_2} f_2$ approximately coincide as $h \to 0$

$$K^h_{X_1} f_1 \simeq K^h_{X_2} f_2 \tag{2.27}$$

under condition

$$f_2 = V(X_2 \leftarrow X_1) f_1, \qquad X_2 \sim X_1$$

for some unitary operator $V(X_2 \leftarrow X_1)$. Let us investigate its properies. It is obvious that

$$\begin{gathered} X_1 \sim X_2, X_2 \sim X_3 \Rightarrow X_1 \sim X_3; \\ V(X_3 \leftarrow X_1) = V(X_3 \leftarrow X_2) V(X_2 \leftarrow X_1). \end{gathered} \tag{2.28}$$

Further, it follows from eq.(2.27) that $\mathcal{U}^h_g K^h_{X_1} f_1 \simeq \mathcal{U}^h_g K^h_{X_2} f_2$, so that

$$\begin{gathered} X_1 \sim X_2 \Rightarrow u_g X_1 \sim u_g X_2; \\ V(u_g X_2 \leftarrow u_g X_1) U_g(u_g X_1 \leftarrow X_1) = U_g(u_g X_2 \leftarrow X_2) V(X_2 \leftarrow X_1). \end{gathered} \tag{2.29}$$

If quantum field operators (such as vector potential) were well-defined for gauge theories, the relation $\sqrt{h}\hat{\varphi}(x) K^h_{X_1} f_1 \simeq \sqrt{h}\hat{\varphi}(x) K^h_{X_2} f_2$ would imply that

$$\begin{gathered} \Phi(x|X_2) = \Phi(x|X_1); \\ \hat{\phi}(x|X_2) V(X_2 \leftarrow X_1) = V(X_2 \leftarrow X_1) \hat{\phi}(x|X_1), \quad X_1 \sim X_2. \end{gathered} \tag{2.30}$$

Finally, let (X_i, f_i) be α-dependent. Let us differentiate relation (2.27); $ih \frac{\partial}{\partial \alpha_a} K^h_{X_1} f_1 \simeq ih \frac{\partial}{\partial \alpha_a} K^h_{X_2} f_2$. One obtains

$$\begin{gathered} \omega_{X_1}[\tfrac{\partial X_1}{\partial \alpha_a}] = \omega_{X_2}[\tfrac{\partial X_2}{\partial \alpha_a}]; \\ V(X_2 \leftarrow X_1) \Omega_{X_1}[\tfrac{\partial X_1}{\partial \alpha_a}] = \Omega_{X_2}[\tfrac{\partial X_2}{\partial \alpha_a}] V(X_2 \leftarrow X_1), \quad X_1(\alpha) \sim X_2(\alpha). \end{gathered} \tag{2.31}$$

In particular, relations (2.31) imply that

$$\omega_X[\delta X]=0, \quad \Omega_X[\delta X]=0 \text{ if } X+\delta X\sim X.$$

Therefore, for gauge theories an additional axiom of semiclassical theory (concerning $V(X_2\leftarrow X_1)$) should be formulated.

9. It happens that axioms A2 and A3 should be revised. Namely, classical states $u_{g_1g_2}X$ and $u_{g_1}u_{g_2}X$ may be equivalent but not equal. This means that property (2.9) should be rewritten as

$$\begin{gathered} u_{g_1g_2}X\sim u_{g_1}u_{g_2}X, \\ U_{g_1g_2}(u_{g_1g_2}X\leftarrow X)=V(u_{g_1g_2}X\leftarrow u_{g_1}u_{g_2}X)U_{g_1}(u_{g_1}u_{g_2}X\leftarrow u_{g_2}X)U_{g_2}(u_{g_2}X\leftarrow X). \end{gathered} \tag{2.32}$$

Axiom A3 also reqiure a revision since the vector potential $\mathcal{A}^\mu(x)$ is *not* an observable. It is more convenient to consider *gauge-invariant* observables

$$\hat{O}=O[\sqrt{h}\hat{\varphi}(\cdot)].$$

An analog of commutation rule (2.7) will be written as

$$\hat{O}K^h_Xf\simeq K^h_X[O(X)+\sqrt{h}\Xi O(X)+...]f. \tag{2.33}$$

We see that for gauge theories one should assign a c-number quantity $O(X)$ and an operator $\Xi O(X)$ to each gauge-invariant functional $O[\Phi(\cdot)]$.

Note that for the scalar case

$$O(X)=O[\Phi(\cdot|X)]; \quad \Xi O(X)=\int dx\frac{\delta O}{\delta\Phi(x)}\hat{\phi}(x|X). \tag{2.34}$$

Investigate general properties of the infinitesimal objects. First of all, write the Poincare invariance property

$$\mathcal{U}^h_{g^{-1}}O[\sqrt{h}\hat{\varphi}(\cdot)]\mathcal{U}^h_g=O[\sqrt{h}v_g\hat{\varphi}(\cdot)]=(v_gO)[\sqrt{h}\hat{\varphi}(\cdot)]. \tag{2.35}$$

Here for scalar and vector fields

$$v_g\hat{\varphi}(\cdot)\equiv\hat{\varphi}(w_g\cdot); \quad v_g\hat{A}^\mu(\cdot)\equiv\Lambda^\mu_\nu\hat{A}^\nu(w_g\cdot); \quad w_gx=\Lambda^{-1}(x-a).$$

Making use of relation (2.33), one finds that

$$\begin{gathered} O(u_gX)=(v_gO)(X); \\ \Xi O(u_gX)U_g(u_gX\leftarrow X)=U_g(u_gX\leftarrow X)\Xi(v_gO)(X). \end{gathered} \tag{2.36}$$

Next, obtain an analog of relation (2.22). Let $X = X(\alpha)$. Apply the differential operator $ih\frac{\partial}{\partial\alpha_a}$ to relation (2.33). making use of eq.(2.13), one obtains

$$[(O+\sqrt{h}\Xi O+...)(X);(\omega-\sqrt{h}\Omega+...)_X[\frac{\partial X}{\partial\alpha_a}]] = ih\frac{\partial}{\partial\alpha_a}(O+\sqrt{h}\Xi O+...)(X).$$

In the leading order in $\sqrt{h}$,

$$[(\Xi O)(X),\Omega_X[\frac{\partial X}{\partial\alpha_a}]] = -i\frac{\partial O(X)}{\partial\alpha_a}. \quad (2.37)$$

Eq.(2.37) can be also rewritten in terms of differential forms:

$$[(\Xi O)(X),\Omega_X[\delta X]] = -idO(\delta X). \quad (2.38)$$

The operator $(\Xi O)(X)$ can be expressed via Ω_X. Namely, if it is looked for in the form

$$(\Xi O)(X) = -\Omega_X[\nabla_O X], \quad (2.39)$$

the comutation relation (2.38) will take the form

$$d\omega(\cdot,\nabla_O X) = dO. \quad (2.40)$$

One should investigate the problem of solvability of eq.(2.40). It happens that the solution $\nabla_O X$ of (2.40) is found up to a vector δX_0 such that $\omega_X[\delta X_0] = 0$. Therefore, the operator $\Omega_X[\nabla_O X]$ is well-defined.

To justify relation (2.39) up to a c-number, one should check that any operator commuting with all $\Omega_X[\delta X]$ is a c-number. This is a correct statement for electrodynamiics.

Finally, obtain an analog of eq.(2.30). Since $\hat{O}K^h_{X_1}f_1 \simeq \hat{O}K^h_{X_2}f_2$ under conditions (2.27), one has

$$O(X_2) = O(X_1); \Xi O(X_2)V(X_2 \leftarrow X_1) = V(X_2 \leftarrow X_1)\Xi O(X_1), \quad X_1 \sim X_2. \quad (2.41)$$

Thus, for gauge theories one should reformulate axioms A2, A3, A4 and formulate a new axiom A5.

A2'. *For each Poincare transformation $g \in G$, a transformation $u_g : \mathcal{X} \to \mathcal{X}$ and an unitary operator $U_g(u_g X \leftarrow X) : \mathcal{F}_X \to \mathcal{F}_{u_g X}$ are given. The property (2.32) is satisfied.*

A3'. *Let $O[\Phi(\cdot)]$ be a gauge-invariant classical functional of fields Φ. Then classical and semiclassical components ($O(X)$ and $\Xi O(X)$) of the quantum observable are given*

for all $X \in \mathcal{X}$. $O(X)$ is a c-number classical observable, $\Xi O(X)$ is an operator in $\mathcal{F}_X$. The property (2.36) of Poincare invariance of observables is satidfied.

A4': eq.(2.22) should be substituted be eq.(2.37).

A5. *An equivalence relation on the base $\mathcal{X}$ is given. For any pair classically equivalent states $X_1 \sim X_2$ an unitary operator $V(X_2 \leftarrow X_1) : \mathcal{F}_{X_1} \to \mathcal{F}_{X_2}$ is specified. It satisfies properties (2.28), (2.29), (2.31) and (2.41).*

Let us check now the expectations of this section for semiclassical gauge theories. Scalar electrodynamics is a simple example of gauge theory. First, review the main approaches to quantize the theory in Hamiltonian, Couloumb and Lorentz gauges. Then the semiclassical approximation will be developed.

3 Quantization of scalar electrodynamics (Hamoltonian, Couloumb and Lorentz gauges)

There are different ways to quantize gauge theories. One can use the Dirac approach [20] or the manifestly covariant BRST-BFV quantization [21, 22]. Let us review these approaches for the scalar electrodynamics - a model specifying interaction of the complex scalar field θ with electromagnetic field A^μ. To simplify notations, set $(A^\mu, \theta, \theta^*) \equiv \varphi$.

3.1 Dirac quantization

1. One starts from the Lagrangian of the form

$$\mathcal{L} = D_\mu \theta^* D^\mu \theta - m^2 \theta^* \theta - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{h} V(h\theta^*\theta). \tag{3.1}$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $D_\mu = \partial_\mu - i\sqrt{h} A_\mu$ is a covariant derivative (electric charge is set to be $\sqrt{h}$ for simplification of notations), A_μ is a vector potential, θ is a scalar field of mass m, V is a self-interaction potential of the scalar field. The momenta canonically conjugated to A^μ, θ and θ^* are

$$E_\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}^\mu} = F_{\mu 0}, \qquad \pi_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}^*} = D_0 \theta, \qquad \pi_\theta^* = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = D_0 \theta^*,$$

so that $E_0 = 0$. The Hamiltonian has the form

$$H = \int d\mathbf{x} [\mathcal{H}(\mathbf{x}) + A_0(\mathbf{x}) \Lambda_{\mathbf{x}}]$$

with the Hamiltonian density

$$\mathcal{H}(\mathbf{x}) = \frac{1}{2}E_kE_k + \frac{1}{4}F_{ij}F_{ij} + \pi^*_\theta\pi_\theta + D_i\theta^*D_i\theta + m^2\theta^*\theta + \frac{1}{h}V(h\theta^*\theta) \tag{3.2}$$

and constraints

$$\Lambda_{\mathbf{x}} = \partial_kE_k + i\sqrt{h}(\pi^*_\theta\theta - \pi_\theta\theta^*). \tag{3.3}$$

The A_0-component of the vector potential appears to be a Lagrange multiplier.

2. Consider the quantum theory in the functional Schrodinger representation. States of the system are specified as functionals $\Psi[A^k(\cdot), \theta(\cdot), \theta^*(\cdot)] \equiv \Psi[\varphi(\cdot)]$. The field operators $\hat{\varphi}(\mathbf{x}) \equiv (\hat{A}^k(\mathbf{x}), \hat{\theta}(\mathbf{x}), \hat{\theta}^*(\cdot))$ are multiplicators by $A^k(\mathbf{x})$, $\theta(\mathbf{x})$ and $\theta^*(\mathbf{x})$, while the momenta operators $\hat{\pi}(\mathbf{x}) \equiv (\hat{E}_k(\mathbf{x}), \hat{\pi}_\theta(\mathbf{x}), \hat{\pi}^*_\theta(\mathbf{x}))$ are

$$\hat{E}_k = -i\frac{\delta}{\delta A^k(\mathbf{x})}, \qquad \hat{\pi}_\theta(\mathbf{x}) = -i\frac{\delta}{\delta\theta^*(\mathbf{x})}, \qquad \hat{\pi}^*_\theta(\mathbf{x}) = -i\frac{\delta}{\delta\theta(\mathbf{x})}. \tag{3.4}$$

The quantum operator $\hat{\mathcal{H}}(\mathbf{x})$ corresponding to the classical Hamiltonian density $\mathcal{H}(\mathbf{x})$ (3.2) is obtained from expression (3.2) by substituting classical variables by their quantum analogs (3.4), while

$$\hat{P}^0 = \int d\mathbf{x}\hat{\mathcal{H}}(\mathbf{x}) \tag{3.5}$$

is quantum Hamiltonian. The Schrodinger equation for the time-dependent states $\Psi^t[A^k(\cdot), \theta(\cdot), \theta^*(\cdot)]$ reads

$$i\dot{\Psi}^t = \hat{P}^0\Psi^t. \tag{3.6}$$

3. There are several ways to take the constraints into account.

In the original Dirac approach [20], physical states Ψ^t_D satisfy not only equation (3.6) but also the additional conditions

$$\hat{\Lambda}_{\mathbf{x}}\Psi^t_D = 0 \tag{3.7}$$

The operators $\hat{\Lambda}_{\mathbf{x}}$ are quantum analogs of constraints (3.3),

$$\hat{\Lambda}_{\mathbf{x}} = \partial_k\frac{1}{i}\frac{\delta}{\delta A^k(\mathbf{x})} + \sqrt{h}\left(\theta(\mathbf{x})\frac{\delta}{\delta\theta(\mathbf{x})} - \theta^*(\mathbf{x})\frac{\delta}{\delta\theta^*(\mathbf{x})}\right). \tag{3.8}$$

Since

$$[\hat{\Lambda}_{\mathbf{x}}; \hat{P}_0] = 0, \tag{3.9}$$

condition (3.7) conserves under time evolution. The most difficult problem in the original Dirac approach is to introduce an inner product.

One can perform the Couloumb gauge quantization. The wave functionals Ψ_C are considered on the surface

$$\partial_k A^k(\mathbf{x}) = 0 \tag{3.10}$$

only. They depend on $A^k_\perp$, θ, θ^* then,

$$\Psi_C = \Psi_C[A_\perp, \theta, \theta^*],$$

where

$$A^k_\perp(\mathbf{x}) = (\delta_{kl} - \frac{\partial_k \partial_l}{\partial^2}) A^l(\mathbf{x}); \tag{3.11}$$

so that

$$A^k(\mathbf{x}) = (\delta_{kl} - \frac{\partial_k \partial_l}{\partial^2}) A^l_\perp(\mathbf{x}) + \partial_k \gamma(\mathbf{x}), \qquad \gamma(\mathbf{x}) = \frac{1}{\partial^2} \partial_l A^l(\mathbf{x}). \tag{3.12}$$

The operators $\hat{A}^k(\mathbf{x})$ and $\hat{E}_k(\mathbf{x})$ should be rewritten in the Couloumb gauge in the following way. Since Ψ is viewed on the surface (3.10), one has $\hat{A}^k(\mathbf{x}) = A^k_\perp(\mathbf{x})$. One also has

$$\frac{1}{i} \frac{\delta}{\delta A^k(\mathbf{x})} = \left(\delta_{kl} - \frac{\partial_k \partial_l}{\partial^2} \right) \frac{1}{i} \frac{\delta}{\delta A^l_\perp(\mathbf{x})} - \frac{1}{\partial^2} \partial_l \frac{1}{i} \frac{\delta}{\delta \gamma(\mathbf{x})},$$

provided that the continuation of Ψ_C for arbitrary A^k is given. If condition (3.7) is satisfied, one has

$$\left[\frac{\delta}{\delta \gamma(\mathbf{x})} + i\sqrt{h} \left(\theta(\mathbf{x}) \frac{\delta}{\delta \theta(\mathbf{x})} - \theta^*(\mathbf{x}) \frac{\delta}{\delta \theta^*(\mathbf{x})} \right) \right] \Psi_C = 0.$$

Therefore, in the Couloumb gauge the field operators are

$$\begin{gathered} \hat{E}_k^{(C)} = \left(\delta_{kl} - \frac{\partial_k \partial_l}{\partial^2} \right) \frac{1}{i} \frac{\delta}{\delta A^l_\perp(\mathbf{x})} - \sqrt{h} \frac{1}{\partial^2} \partial_k \left(\theta(\mathbf{x}) \frac{\delta}{\delta \theta(\mathbf{x})} - \theta^*(\mathbf{x}) \frac{\delta}{\delta \theta^*(\mathbf{x})} \right); \\ \hat{A}^k(\mathbf{x}) = A^k_\perp(\mathbf{x}). \end{gathered} \tag{3.13}$$

The quantum Hamiltonian density $\hat{\mathcal{H}}(\mathbf{x})$ is obtained from expression (3.2) by substituting classical variables by quantum analogs (3.13), while quantum Hamiltonian is of the form (3.5). The inner product in the Couloumb gauge is

$$< \Psi_C | \Psi_C >= \int D A_\perp D\theta^* D\theta |\Psi_C[A_\perp, \theta, \theta^*]|^2. \tag{3.14}$$

4. An alternative way to quantize gauge theories is to use the refined algebraic quantization approach [23]. States will be denoted as Ψ_H ("Hamiltonian gauge"). It is the most

suitable quantization for semiclassical approximation. Instead of imposing the constraints on physical states, one modifies the inner product of the theory [24],

$$\begin{array}{c} <\Psi_H,\Psi_H>=(\Psi_H,\prod_{\mathbf{x}}\delta(\hat{\Lambda}_{\mathbf{x}})\Psi_H)= \\ \int DAD\theta D\theta^*(\Psi_H[A,\theta,\theta^*])^*\prod_{\mathbf{x}}\delta(\Lambda_{\mathbf{x}})\Psi_H[A,\theta,\theta^*]. \end{array} \tag{3.15}$$

Because of eq.(3.9), the inner product (3.15) is invariant under time evolution.

Note that the inner product (3.15) is degenerate. For example, states of the form

$$\int d\mathbf{x}\alpha(\mathbf{x})\hat{\Lambda}_{\mathbf{x}}Y; \qquad (\exp(\frac{i}{\sqrt{h}}\int d\mathbf{x}\alpha(\mathbf{x})\hat{\Lambda}_{\mathbf{x}})-1)Y \tag{3.16}$$

are of zero norm. Thus, we should say that state functionals Ψ_H and Ψ'_H are equivalent if their difference is of zero norm,

$$\Psi_H\sim\Psi'_H\equiv<\Psi_H-\Psi'_H,\Psi_H-\Psi'_H>=0.$$

The corresponding factorspace is viewed as a physical state space. Thus, quantum states

$$\Psi_H\sim\exp(\frac{i}{\sqrt{h}}\int d\mathbf{x}\alpha(\mathbf{x})\hat{\Lambda}_{\mathbf{x}})\Psi_H \tag{3.17}$$

may be viewed as gauge equivalent.

Relationship between states Ψ_H and Ψ in the Dirac and refined algebraic quantization approaches is as follows (cf. [25]),

$$\Psi_D=\prod_{\mathbf{x}}\delta(\hat{\Lambda}_{\mathbf{x}})\Psi_H. \tag{3.18}$$

We notice that condition (3.7) is automatically satisfied, while equivalent Ψ_H-states give the same Ψ-state.

An explicit form of the operator $\prod_{\mathbf{x}}\delta(\hat{\Lambda_{\mathbf{x}}})$ can be written via the following functional integral,

$$\prod_{\mathbf{x}}\delta(\hat{\Lambda}_{\mathbf{x}})=\int D\alpha\exp[-\frac{i}{\sqrt{h}}\int d\mathbf{x}\alpha(\mathbf{x})\hat{\Lambda}_{\mathbf{x}}]. \tag{3.19}$$

One also has

$$\begin{array}{c} \exp[-\frac{i\tau}{\sqrt{h}}\int d\mathbf{x}\alpha(\mathbf{x})\Lambda_{\mathbf{x}}]\Psi_H[A^k(\cdot),\theta(\cdot),\theta^*(\cdot)]= \\ \Psi_H[A^k(\cdot)+\frac{\tau}{\sqrt{h}}\partial_k\alpha(\cdot),\theta(\cdot)e^{-i\tau\alpha(\cdot)},\theta^*(\cdot)e^{i\tau\alpha(\cdot)}], \end{array}$$

since both left-hand and right-hand sides of this relation obey the same equation

$$i\sqrt{h}\frac{\partial\Psi_H^\tau}{\partial\tau}=\int d\mathbf{x}\alpha(\mathbf{x})\hat{\Lambda}_{\mathbf{x}}\Psi_H^\tau$$

and initial condition. We see that operator (3.19) generates a gauge transformation on the configuration field space. Therefore, formulas (3.15), (3.18) can be written as

$$(\Psi_H,\Psi_H)=\int DAD\theta^* D\theta D\alpha \Psi^*_H[A,\theta,\theta^*]\Psi_H[A+\frac{1}{\sqrt{h}}\partial\alpha,\theta e^{-i\alpha},\theta^* e^{i\alpha}]; \tag{3.20}$$

$$\Psi_D[A,\theta,\theta^*]=\int D\alpha\Psi_H[A+\frac{1}{\sqrt{h}}\partial\alpha,\theta e^{-i\alpha},\theta^* e^{i\alpha}]. \tag{3.21}$$

Let us perform the linear change of variables (3.12) and consider the functional $\Psi_D[A_\perp,\gamma,\theta,\theta^*]$. Then the obtained formulas will be taken to the form

$$\begin{array}{c}\Psi_D[A_\perp,\gamma,\theta,\theta^*]=\int D\alpha\Psi_H[A_\perp,\gamma+\frac{1}{\sqrt{h}}\alpha,\theta e^{-i\alpha},\theta^* e^{i\alpha}];\\ <\Psi_H|\Psi_H>=\int DA_\perp D\gamma D\theta^* D\theta D\alpha \Psi^*_H[A_\perp,\gamma,\theta^*,\theta]\Psi_H[A_\perp,\gamma+\frac{1}{\sqrt{h}}\alpha,\theta e^{-i\alpha},\theta^* e^{i\alpha}]=\\ \int DA_\perp D\theta^* D\theta|\Psi[A_\perp,0,\theta,\theta^*]|^2.\end{array}$$

Formula (3.14) is then justified.

5. Any Poincare transformation (a,Λ),

$$x^{\mu\prime}=\Lambda^\mu_\nu x^\nu+a^\mu$$

is a composition of time and space translations, boost and spatial rotations,

$$(a,\Lambda)=(a^0,1)(\mathbf{a},1)(0,\exp(\alpha^k l^{0k}))(0,\exp(\frac{1}{2}\theta_{sm}l^{sm})) \tag{3.22}$$

with $\theta_{sm}=-\theta_{ms}$,

$$(l^{\lambda\mu})^\alpha_\beta=-g^{\lambda\alpha}\delta^\mu_\beta+g^{\mu\alpha}\delta^\lambda_\beta.$$

The operator $\hat{\mathcal{U}}^h_{a,\Lambda}$ of the quantum Poincare transformation is

$$\hat{\mathcal{U}}^h_{a,\Lambda}=\exp[i\hat{P}^0a^0]\exp[-i\hat{P}^ja^j]\exp[i\alpha^k\hat{M}^{0k}]\exp[\frac{i}{2}\hat{M}^{lm}\theta_{lm}]. \tag{3.23}$$

The operator $\hat{P}^0$ has been already defined (formula (3.5)), while

$$\begin{array}{c}\hat{P}^l=\int d\mathbf{x}\hat{\mathcal{P}}^l(\mathbf{x});\quad \hat{M}^{k0}=\int d\mathbf{x}x^k\hat{\mathcal{H}}(\mathbf{x});\\ \hat{M}^{kl}=\int d\mathbf{x}[x^k\hat{\mathcal{P}}^l(\mathbf{x})-x^l\hat{\mathcal{P}}^k(\mathbf{x})+\hat{E}_l\hat{A}^k-\hat{E}_k\hat{A}^l].\end{array} \tag{3.24}$$

Here the operators $\hat{\mathcal{H}}(\mathbf{x})$, $\hat{\mathcal{P}}^l(\mathbf{x})$, $\hat{E}_l$, $\hat{A}^l$ are obtained from eq.(3.2) and expression

$$\mathcal{P}^l=-\partial_l\theta\pi^*-\partial_l\theta^*\pi-\partial_lA^sE_s$$

by substituting classical variables by their quantum analogs, eqs.(3.4) and (3.13) for Hamiltonian and Couloumb gauges correspondingly.

Since Poincare generators commute with constraints on the constraint surface, condition (3.7), inner product (3.15) conserve under Poincare transformations. The Poincare algebra infinitesimal relations

$$
\begin{gathered}
[\hat{P}^\lambda, \hat{P}^\mu] = 0, \quad \left[\hat{M}^{\lambda\mu}, \hat{P}^\sigma\right] = i(g^{\mu\sigma}\hat{P}^\lambda - g^{\lambda\sigma}\hat{P}^\mu); \\
\left[\hat{M}^{\lambda\mu}, \hat{M}^{\rho\sigma}\right] = -i(g^{\lambda\rho}\hat{M}^{\mu\sigma} - g^{\lambda\sigma}\hat{M}^{\mu\rho} + g^{\mu\sigma}\hat{M}^{\lambda\rho} - g^{\mu\rho}\hat{M}^{\lambda\sigma})
\end{gathered}
\tag{3.25}
$$

are satisfied on the constraint surface. This implies that the operators $\hat{\mathcal{U}}^h_{a,\Lambda}$ indeed form a representation of the Poincare group.

3.2 The Gupta-Bleuler (BRST-BFV) approach

The manifestly covariant quantization technique [21, 22] (quantization in Lorentz gauge) of electrodynamics is as follows (see, for example, [26]). States are specified as functionals

$$\Psi_L[A^0(\cdot), A^k(\cdot), \theta(\cdot), \theta^*(\cdot)]. \tag{3.26}$$

An indefinite inner product is introduced:

$$
\begin{gathered}
< \Psi_L, \Psi_L >= \\
\int DA^k D\lambda D\theta D\theta^* (\Psi_L[A^k(\cdot), -i\lambda(\cdot), \theta(\cdot), \theta^*(\cdot)])^* \Psi_L[A^k(\cdot), i\lambda(\cdot), \theta(\cdot), \theta^*(\cdot)]
\end{gathered}
\tag{3.27}
$$

The Gupta-Bleuler constraint condition is imposed on physical states:

$$[\frac{1}{i}\frac{\delta}{\delta A^0(\mathbf{x})} - \frac{i}{\sqrt{-\Delta}}\Lambda_{\mathbf{x}}]\Psi_L = 0. \tag{3.28}$$

Moreover, states

$$\Psi_L \sim \Psi_L + \int d\mathbf{x}\beta(\mathbf{x})[\frac{1}{i}\frac{\delta}{\delta A^0(\mathbf{x})} + \frac{i}{\sqrt{-\Delta}}\Lambda_{\mathbf{x}}]Y_L \tag{3.29}$$

are set to be equivalent. Definition of equivalence realtion (3.29) is reasonable since state $\int d\mathbf{x}\beta(\mathbf{x})[\frac{1}{i}\frac{\delta}{\delta A^0(\mathbf{x})} + \frac{i}{\sqrt{-\Delta}}\Lambda_{\mathbf{x}}]Y_L$ is orthogonal to any physical state.

One can notice that this approach is equivalent to the Dirac approach. Condition (3.28) implies that

$$\Psi_L[A^k(\cdot), A^0(\cdot), \theta(\cdot), \theta^*(\cdot)] = \exp[-\int d\mathbf{x} A^0(\mathbf{x})\frac{1}{\sqrt{-\Delta}}\Lambda_{\mathbf{x}}]\Psi_H[A^k(\cdot), \theta(\cdot), \theta^*(\cdot)]. \tag{3.30}$$

The inner product (3.27) will be rewritten then as

$$\int DA^k D\theta D\theta^* \Psi_H^*[A^k(\cdot),\theta(\cdot),\theta^*(\cdot)] \int D\lambda e^{-2i\int d\mathbf{x}\lambda_{\mathbf{x}}\frac{1}{\sqrt{-\Delta}}\Lambda_{\mathbf{x}}} \Psi_H[A^k(\cdot),\theta(\cdot),\theta^*(\cdot)]$$

We come to formula (3.15) up to a field-independent normalizing factor.

The Poincare generators are given by formulas (3.5), (3.24) with

$$\hat{\mathcal{H}}_L = \hat{\mathcal{H}} + \hat{A}_0\hat{\Lambda}_{\mathbf{x}} - \frac{\xi}{2}\hat{E}_0^2 + \hat{A}^k\partial_k\hat{E}_0,$$
$$\hat{\mathcal{P}}_L^l = \hat{\mathcal{P}}^l - \partial_l\hat{A}^0\hat{E}_0.$$

Here $\hat{\mathcal{H}}$ and $\hat{\mathcal{P}}^l$ are Hamiltonian and momenta densities for the Hamiltonian gauge, $\hat{E}_0(\mathbf{x}) = \frac{1}{i}\frac{\delta}{\delta A^0(\mathbf{x})}$, ξ is a real parameter. The algebraic properties (3.25) are satisfied exactly (not only on the constraint surface). Making use of eq.(3.30), we find that equations of motion for Lorentz and Hamiltonian gauges are in agreement.

4 Semiclassical states

In section 2 we considered "elementary" semiclassical states (2.5) and their superpositions (2.6) for scalar field models. Let us now write their analogs for the scalar electrodynamics in Hamiltonian, Couloumb and Lorentz gauges and investigate their properties.

4.1 Semiclassical states in refined algebraic quantization approaches (Hamiltonian gauge)

Consider state (2.5) in Hamiltonian gauge

$$\Psi_H[\varphi(\cdot)] = e^{\frac{i}{h}S} e^{\frac{i}{h}\int d\mathbf{x}\Pi(\mathbf{x})(\varphi(\mathbf{x})\sqrt{h}-\Phi(\mathbf{x}))} g[\varphi(\cdot) - \frac{\Phi(\cdot)}{\sqrt{h}}] \equiv K_X^h g[\varphi(\cdot)]. \tag{4.1}$$

For the simplicity, the following notations are introduced:

$$\varphi \equiv (A^k,\theta,\theta^*), \quad \Phi \equiv (\mathcal{A}^k,\Theta,\Theta^*), \quad \Pi \equiv (\mathcal{E}_k,\Pi_{theta},\Pi_\theta^*);$$

φ is field configuration, Φ are classical fields, Π are classical momenta; for integrals, the followig simplification is used:

$$\int d\mathbf{x}\Pi\Phi \equiv \int d\mathbf{x}[\mathcal{E}_k\mathcal{A}^k + \Pi_\theta\Theta^* + \Pi_\theta^*\Theta]; \quad \int d\mathbf{x}\Pi\varphi \equiv \int d\mathbf{x}[\mathcal{E}_kA^k + \Pi_\theta\theta^* + \Pi_\theta^*\theta].$$

Introduce a special notation for a gauge rransformation in the configuration space:

$$\nu_\alpha\varphi \equiv \nu_\alpha(A,\theta,\theta^*) = (A + \frac{1}{\sqrt{h}}\partial\alpha, \theta e^{-i\alpha}, \theta^* e^{i\alpha}).$$

Then, the inner product (3.20) will be written as

$$(\Psi_H,\Psi_H) = \int D\alpha D\varphi e^{\frac{i}{\sqrt{h}}\int d\mathbf{x}\Pi(\mathbf{x})(\nu_\alpha\varphi(\mathbf{x})-\varphi(\mathbf{x}))} g^*[\varphi - \frac{\Phi}{\sqrt{h}}]g[\nu_\alpha\varphi - \frac{\Phi}{\sqrt{h}}] \tag{4.2}$$

Here the integration measure

$$D\varphi \equiv DAD\theta^* D\theta.$$

Notice that quantities g and g^* entering to expression (4.2) are not exponentially small only if

$$\varphi - \frac{\Phi}{\sqrt{h}} \sim O(1), \quad \nu_\alpha\varphi - \frac{\Phi}{\sqrt{h}} \sim O(1).$$

Therefore, $\nu_\alpha\varphi - \varphi \sim O(1)$ and $\alpha \sim \sqrt{h}$. Only functions α of such order give a significant contribution to the integral (4.2). To calculate the inner product as $h \to 0$, perform a substitution

$$\alpha = \sqrt{h}\beta, \quad \varphi - \frac{\Phi}{\sqrt{h}} = \phi \equiv (a^k, \vartheta, \vartheta^*).$$

Then the expresiion (4.2) will be taken to the form

$$(\Psi_H,\Psi_H) = \int D\beta D\phi e^{\frac{i}{\sqrt{h}}\int d\mathbf{x}\Pi(\mathbf{x})(\nu_{\beta\sqrt{h}}(\frac{\Phi}{\sqrt{h}}+\phi)(\mathbf{x}) - (\frac{\Phi}{\sqrt{h}}+\phi)(\mathbf{x}))} g^*[\phi]g[\nu_{\beta\sqrt{h}}(\frac{\Phi}{\sqrt{h}}+\phi) - \frac{\Phi}{\sqrt{h}}] \tag{4.3}$$

Let us evaluate the expressions entering to the inner product (4.3). One has

$$\begin{gathered}\nu_{\beta\sqrt{h}}(\frac{\Phi}{\sqrt{h}}+\phi) = \nu_{\beta\sqrt{h}}\left(\frac{\mathcal{A}}{\sqrt{h}}+a, \frac{\Theta}{\sqrt{h}}+\vartheta, \frac{\Theta^*}{\sqrt{h}}+\vartheta^*\right) \\ = \left(\frac{\mathcal{A}}{\sqrt{h}}+a+\partial\beta, \left[\frac{\Theta}{\sqrt{h}}+\vartheta\right]e^{-i\beta\sqrt{h}}, \left[\frac{\Theta^*}{\sqrt{h}}+\vartheta^*\right]e^{i\beta\sqrt{h}}\right)\end{gathered}$$

Therefore,

$$\nu_{\beta\sqrt{h}}(\frac{\Phi}{\sqrt{h}}+\phi) - (\frac{\Phi}{\sqrt{h}}+\phi) = \left(\partial\beta, \left[\frac{\Theta}{\sqrt{h}}+\vartheta\right](e^{-i\beta\sqrt{h}}-1), \left[\frac{\Theta^*}{\sqrt{h}}+\vartheta^*\right](e^{i\beta\sqrt{h}}-1)\right),$$

so that

$$\begin{gathered}\int d\mathbf{x}\Pi[\nu_{\beta\sqrt{h}}(\frac{\Phi}{\sqrt{h}}+\phi) - (\frac{\Phi}{\sqrt{h}}+\phi)] = \\ \int d\mathbf{x}\left\{\mathcal{E}_k\partial_k\beta + \Pi^*_\theta\left[\frac{\Theta}{\sqrt{h}}+\vartheta\right](e^{-i\beta\sqrt{h}}-1) + \Pi_\theta\left[\frac{\Theta^*}{\sqrt{h}}+\vartheta^*\right](e^{i\beta\sqrt{h}}-1)\right\}\end{gathered}$$

Notice that the integrand in (4.3) is a product of a slowly varying and damping at the infinity functional by the rapidly oscillatng exponent

$$\exp\left\{-\frac{i}{\sqrt{h}}\int d\mathbf{x}\beta(\mathbf{x})(\partial_k\mathcal{E}_k + i(\Pi_\theta^*\Theta - \Pi_\theta\Theta^*))\right\}.$$

Therefore, the integral will be exponentially small, except for the case

$$\Lambda_{\mathbf{x}} \equiv \partial_k\mathcal{E}_k + i(\Pi_\theta^*\Theta - \Pi_\theta\Theta^*) = 0. \tag{4.4}$$

An analogous fact was discovered in section 2: it was found that the "composed state" (2.6) is exponentially small if the Maslov isotropic condition is not satisfied. Now we see that for constrained systems additional conditions arise even for wave packet states (4.1).

Under condition (4.4), one can simplify expression (4.3) as $h \to 0$. One should write

$$g\left[\nu_{\beta\sqrt{h}}(\tfrac{\Phi}{\sqrt{h}} + \phi) - (\tfrac{\Phi}{\sqrt{h}} + \phi)\right] \simeq g(a + \partial\beta, \vartheta - i\beta\Theta, \vartheta^* + i\beta\Theta^*) = e^{\int d\mathbf{x}[\partial\beta\frac{\delta}{\delta a} - i\beta\Theta\frac{\delta}{\delta\vartheta} + i\beta\Theta^*\frac{\delta}{\delta\vartheta^*}]}g(a, \vartheta, \vartheta^*),$$

use the Baker-Hausdorff formula for exponents and obtain that

$$(\Psi_H, \Psi_H) = \int D\phi g^*[\phi]\int D\beta e^{-i\int d\mathbf{x}\beta(\mathbf{x})(\Xi\Lambda_{\mathbf{x}})}g[\phi] = (g, \prod_{\mathbf{x}}\delta(\Xi\Lambda_{\mathbf{x}})g), \tag{4.5}$$

where

$$\Xi\Lambda_{\mathbf{x}} \equiv -i\partial_k\frac{\delta}{\delta a^k} + i\Pi_\theta^*\vartheta + \Theta\frac{\delta}{\delta\vartheta} - i\Pi_\theta\vartheta^* - \Theta^*\frac{\delta}{\delta\vartheta^*} \tag{4.6}$$

be a linearized constraint (4.4). Since

$$[\Xi\Lambda_{\mathbf{x}}, \Xi\Lambda_{\mathbf{y}}] = 0,$$

there are no operator ordering problems in (4.5).

Thus, there are the following new features of semiclassical electrodynamics due to gauge symmetry.

First, not any classical configuration $X = (S, \Pi, \Phi)$ can be chosen: the classical constraint condition (4.4) should be satisfied; otherwise, state $K_X^h f$ will have zero norm. Therefore, the base of the semiclassical bundle (classical state space) is a "curved" constraint surface in the flat space.

Next, the inner products in fibres $\mathcal{F}_X$ is X-dependent since the linearized constraints $\Xi\Lambda_{\mathbf{x}}$ (4.6) depend on X. The degenerate inner product (4.5) resembles (2.20). One should

consider usual procedures of factorization and completefication of a pre-Hilbert space with inner product (4.5).

The 1-forms ω and Ω have standard forms:

$$\begin{array}{c} \omega_X[\delta X] = \int d\mathbf{x}\Pi(\mathbf{x})\delta\Phi(\mathbf{x}) - \delta S; \\ \Omega_X[\delta X] = \int d\mathbf{x}\left[\delta\Pi(\mathbf{x})\phi(\mathbf{x}) - \delta\Phi(\mathbf{x})\frac{1}{i}\frac{\delta}{\delta\phi(\mathbf{x})}\right]. \end{array} \tag{4.7}$$

The commutation relations (2.15) are satisfied.

It is necessary to check that the operator $\Omega_X[\delta X]$ conserves the equivalence property; it should take zero-norm states to zero-norm states. To justify this property, one should prove that

$$[\Xi\Lambda_{\mathbf{x}}; \Omega_X[\delta X]] = 0. \tag{4.8}$$

Notice that

$$\int d\mathbf{x}\beta(\mathbf{x})\Xi\Lambda_{\mathbf{x}} \equiv -\Xi\Lambda[\beta] = -\Omega_X[\nabla_{\Lambda[\beta]}X],$$

where the infinitesimal vector $\nabla_{\Lambda[\beta]}X \equiv (\nabla_{\Lambda[\beta]}S = 0, \nabla_{\Lambda[\beta]}\Pi, \nabla_{\Lambda[\beta]}\Phi)$ has the form of infinitesimal gauge transformation:

$$\begin{array}{c} \nabla_{\Lambda[\beta]}\mathcal{E}_k = 0, \quad \nabla_{\Lambda[\beta]}\Pi_\theta = i\beta\Pi_\theta, \quad \nabla_{\Lambda[\beta]}\Pi^*_\theta = -i\beta\Pi^*_\theta, \\ \nabla_{\Lambda[\beta]}\mathcal{A}^k = -\partial_k\beta; \quad \nabla_{\Lambda[\beta]}\Theta = i\beta\Theta, \quad \nabla_{\Lambda[\beta]}\Theta^* = -i\beta\Theta^* \end{array} \tag{4.9}$$

It follows from eq.(2.16) and formula for $d\omega_X$ that

$$[\Xi\Lambda[\beta]; \Omega_X[\delta X]] = id\omega_X(\nabla_{\Lambda[\beta]}X; \delta X) = -id\Lambda[\beta](\delta X)$$

with

$$\Lambda[\beta] \equiv \int d\mathbf{x}\beta(\mathbf{x})\Lambda_{\mathbf{x}}. \tag{4.10}$$

Notice that X and $X + \delta X$ should both satisfy the additional condition (4.4) $\Lambda[\beta] = 0$; therefore, quantity (4.10) vanishes.

Let us investigate properties of the 1-form Ω. First of all, notice that

$$\Omega_X[\nabla_{\Lambda[\beta]}X]g = -\int d\mathbf{x}\beta(\mathbf{x})\Xi\Lambda_{\mathbf{x}}g \sim 0 \tag{4.11}$$

for all g since state (4.11) is orthogonal to all states because of relation $\Xi\Lambda_{\mathbf{x}}\prod_{\mathbf{y}}\delta(\Xi\Lambda_{\mathbf{y}}) = 0$. It also happens that the inverse statement is also valid:

$$\Omega_X[\delta X] \sim 0 \quad \Rightarrow \quad \delta X = (\delta S, \delta\Pi = \nabla_{\Lambda[\beta]}\Pi, \delta\Phi = \nabla_{\Lambda[\beta]}\Phi). \tag{4.12}$$

To check implication (4.12), notice that $\Omega_X[\delta X] \sim 0$ implies that $[\Omega_X[\delta X]; \Omega_X[\delta X']] = 0$ for all infinitesimal $\delta X'$, i.e.

$$d\omega_X(\delta X, \delta X') = \int d\mathbf{x}(\delta\mathcal{E}_k\delta\mathcal{A}^{k\prime} + \delta\Pi_\theta\delta\Theta^{*\prime} + \delta\Pi_\theta^*\delta\Theta' - \delta\mathcal{A}^k\delta\mathcal{E}_k' - \delta\Theta\delta\Pi_\theta^{*\prime} - \delta\Theta^*\delta\Pi_\theta') = 0 \tag{4.13}$$

for $\delta X'$ satisfying the constraint relation

$$\partial_k\delta\mathcal{E}_k' + i\delta(\Pi_\theta^{*\prime} - \Pi_\theta'\Theta^{*\prime}) = 0. \tag{4.14}$$

Relation (4.13) implies that $\delta\mathcal{E}_k = 0$, $\delta\mathcal{A}_\perp^k = 0$ since $\delta\mathcal{A}^{k\prime}$ and $\delta\mathcal{E}_{k\perp}'$ may be arbitrary. Therefore, $\delta\mathcal{A}^k = -\partial_k\beta$; making use of (4.14), we check statement (4.12).

For the following sections, it will be necessary to solve the equation

$$d\omega_X(\delta X, \delta X') = \kappa(\delta X'), \quad \delta X-? \tag{4.15}$$

where κ is a given 1-form. It happens that problem (4.15) has a solution iff

$$\kappa(\nabla_{\Lambda[\beta]}X) = 0. \tag{4.16}$$

Namely, implication (4.15) $\to$ (4.16) is evident. To check implication (4.16) $\to$ (4.15), notice that a general form of κ can be written as

$$\kappa(\delta X') = \int d\mathbf{x}(\delta\mathcal{E}_k\delta\mathcal{A}^{k\prime} + \delta\Pi_\theta\delta\Theta^{*\prime} + \delta\Pi_\theta^*\delta\Theta' - \delta\mathcal{A}^k\delta\mathcal{E}_k' - \delta\Theta\delta\Pi_\theta^{*\prime} - \delta\Theta^*\delta\Pi_\theta') = 0$$

For $\delta X = (\delta S, \delta\mathcal{E}_k, \delta\Pi_\theta, \delta\Pi_\theta^*, \delta\mathcal{A}^k, \delta\Theta, \delta\Theta^*)$, eq.(4.16) implies that $d\Lambda[\beta](\delta X) = 0$. Thus, (4.15) $\equiv$ (4.16). Note that the solution of problem (4.15) is not unique: one can add to δX vector $(\delta S, \delta\Pi = \nabla_{\Lambda[\beta]}\Pi, \delta\Phi = \nabla_{\Lambda[\beta]}\Phi)$.

4.2 Dirac semiclassical states. Couloumb gauge

Let us rewrite state (4.1) in the Dirac approach. Making use of relation (3.21), one finds:

$$\Psi_D[\varphi(\cdot)] = \int D\alpha e^{\frac{i}{h}S} e^{\frac{i}{\sqrt{h}}\int d\mathbf{x}\Pi(\nu_\alpha\varphi - \frac{\Phi}{\sqrt{h}})} g[\nu_\alpha\varphi - \frac{\Phi}{\sqrt{h}}]. \tag{4.17}$$

Integral (4.17) is not exponentially small only if $\nu_{\overline{\alpha}}\varphi - \Phi/\sqrt{h} \sim O(1)$ for some $\overline{\alpha}$, i.e.

$$a^k = A^k - \frac{\mathcal{A}^k - \partial_k\overline{\alpha}}{\sqrt{h}} \sim O(1), \quad \vartheta = \theta e^{-i\overline{\alpha}} - \frac{\Theta}{\sqrt{h}} \sim O(1). \tag{4.18}$$

Denote $\phi \equiv (a^k, \vartheta, \vartheta^*)$ and perform a substitution $\alpha = \overline{\alpha} + \sqrt{h}\beta$. One has:

$$\nu_{\overline{\alpha}+\sqrt{h}\beta}\varphi = \left(\frac{\mathcal{A}}{\sqrt{h}} + a + \partial\beta, \left(\frac{\Theta}{\sqrt{h}} + \vartheta\right) e^{-i\sqrt{h}\beta}, \left(\frac{\Theta^*}{\sqrt{h}} + \vartheta^*\right) e^{i\sqrt{h}\beta}\right)$$

and

$$\begin{array}{c} \int d\mathbf{x}\Pi(\nu_\alpha\varphi - \frac{\Phi}{\sqrt{h}}) = \\ \int d\mathbf{x}[\mathcal{E}_k(a^k + \partial_k\beta) + \Pi_\theta^* \frac{\Theta}{\sqrt{h}}(e^{-i\beta\sqrt{h}} - 1) + \Pi_\theta \frac{\Theta^*}{\sqrt{h}}(e^{i\beta\sqrt{h}} - 1) + \Pi_\theta^*\vartheta e^{-i\beta\sqrt{h}} + \Pi_\theta\vartheta^* e^{i\beta\sqrt{h}}] \end{array}$$

Under condition (4.4), one finds analogously to the previous subsection that

$$\begin{array}{c} \Psi_D[\varphi(\cdot)] = e^{\frac{i}{h}S} e^{\frac{i}{\sqrt{h}}\int d\mathbf{x}\Pi(\mathbf{x})\phi(\mathbf{x})} f[\phi(\cdot)]; \\ f[\phi(\cdot)] = \prod_{\mathbf{x}} \delta(\Xi\lambda_{\mathbf{x}}) g[\phi(\cdot)]. \end{array} \tag{4.19}$$

For the Couloumb-gauge quantization, we are interested only in values of the Dirac functional Ψ_D on the surface $\partial_k\mathcal{A}^k = 0$. Therefore, the gauge function $\overline{\alpha}$ should be chosen in such a way that $\partial_k(\mathcal{A}^k - \partial_k\overline{\alpha}) = 0$.

Without loss of generality, one can specify classical states by sets $X = (S, \mathcal{E}_k^\perp, \mathcal{A}_\perp^k, \Theta, \Pi_\theta)$. Then the semiclassical state (4.1) will be written as

$$\Psi_C[\varphi_\perp(\cdot)] = e^{\frac{i}{h}S} e^{\frac{i}{h}\int d\mathbf{x}\Pi_\perp(\mathbf{x})(\varphi_\perp(\mathbf{x})\sqrt{h} - \Phi_\perp(\mathbf{x})} f[\varphi_\perp(\cdot) - \frac{\Phi_\perp(\cdot)}{\sqrt{h}}];$$

here

$$\varphi_\perp \equiv (A_\perp^k, \theta, \theta^*), \quad \Phi_\perp \equiv (\mathcal{A}_\perp^k, \Theta, \Theta^*), \quad \Pi_\perp \equiv (\mathcal{E}_k^\perp, \Pi_\theta, \Pi_\theta^*).$$

One has

$$(\Psi_C, \Psi_C) = \int D\phi_\perp |f[\phi_\perp]|^2 \tag{4.20}$$

for the inner product. It is possible to check that eqs.(4.5), (4.19) and (4.20) indeed do not contradict each other.

The 1-forms are written analogously to (4.7):

$$\begin{array}{c} \omega_X[\delta X] = \int d\mathbf{x}\Pi_\perp(\mathbf{x})\delta\Phi_\perp(\mathbf{x}) - \delta S; \\ \Omega_X[\delta X] = \int d\mathbf{x}[\delta\Pi_\perp(\mathbf{x})\phi_\perp(\mathbf{x}) - \delta\Phi_\perp(\mathbf{x})\frac{1}{i}\frac{\delta}{\delta\phi_\perp(\mathbf{x})}. \end{array}$$

4.3 Semiclassical states in the Gupta-Bleuler approach (Lorentz gauge)

Consider the semiclassical states in the Gupta-Bleuler approach. Let $\mathcal{A}^0(\mathbf{x})$ be some field configuration. Under condition

$$A^0(\mathbf{x}) - \frac{\mathcal{A}^0(\mathbf{x})}{\sqrt{h}} \equiv a^0(\mathbf{x}) \sim O(1),$$

suppose the Gupta-Bleuler state Ψ_L be as follows

$$\Psi_L[\varphi_L(\cdot)] = e^{\frac{i}{h}S} e^{\frac{i}{h}\int d\mathbf{x}\Pi_L(\mathbf{x})(\varphi_L(\mathbf{x})\sqrt{h}-\Phi_L(\mathbf{x}))} v[\varphi_L - \frac{\Phi_L}{\sqrt{h}}].$$

Here

$$\varphi_L \equiv (A^0, A^k, \theta, \theta^*), \quad \Phi_L = (\mathcal{A}^0, \mathcal{A}^k, \Theta, \Theta^*), \quad \Pi_L = (\mathcal{E}_0, \mathcal{E}_k, \Pi_\theta, \Pi_\theta^*).$$

The values of Ψ_L for arbitrary A^0 can be reconstructed from the Gupta-Bleuler condition (3.28). The inner product has the following form:

$$(\Psi_L, \Psi_L) = \int D\lambda Da^k D\vartheta D\vartheta^* (v(-i\lambda, a, \vartheta, \vartheta^*))^* v(i\lambda, a, \vartheta, \vartheta^*). \tag{4.21}$$

Condition (3.28) implies property (4.4), relation $\mathcal{E}_0 = 0$ and

$$\left(\frac{1}{i}\frac{\delta}{\delta a^0(\mathbf{x})} - \frac{i}{\sqrt{-\Delta}} \Xi \Lambda_{\mathbf{x}} \right) v(a^0, a^k, \vartheta, \vartheta^*) = 0. \tag{4.22}$$

Consider the equivalence transformation (3.29) in the semiclassical theory. For $Y_L = K_X^h \zeta$, one finds that

$$v \sim v + \int d\mathbf{x} \beta(\mathbf{x}) \left(\frac{1}{i}\frac{\delta}{\delta a^0(\mathbf{x})} + \frac{i}{\sqrt{-\Delta}} \Xi \Lambda_{\mathbf{x}} \right) \zeta.$$

It follows from eq.(4.22) that

$$v(a^0, a^k, \vartheta, \vartheta^*) = e^{-\int d\mathbf{x} a^0(\mathbf{x}) \frac{1}{\sqrt{-\Delta}} \Xi \Lambda_{\mathbf{x}}} g[a^k, \vartheta, \vartheta^*]$$

with

$$g[a^k, \vartheta, \vartheta^*] = v[0, a^k, \vartheta, \vartheta^*].$$

The inner product (4.21) is then in agreement with (4.5).

The 1-forms ω and Ω have the standard forms:

$$\omega_X[\delta X] = \int d\mathbf{x}\Pi_L(\mathbf{x})\delta\Phi_L(\mathbf{x}) - \delta S;$$
$$\Omega_X[\delta X] = \int d\mathbf{x}\left[\delta\Pi_L(\mathbf{x})\phi_L(\mathbf{x}) - \delta\Phi_L(\mathbf{x})\frac{1}{i}\frac{\delta}{\delta\phi_L(\mathbf{x})}\right]$$

One should check that the operators $\Omega_X[\delta X]$ conserve the additional condition (4.22). It is sufficient to justify that

$$\left[\frac{1}{i}\frac{\delta}{\delta a^0(\mathbf{x})}; \Omega_X[\delta X]\right] = 0, \quad [\Xi\Lambda[\beta]; \Omega_X[\delta X]] = 0.$$

The first property means that $\delta\mathcal{E}_0 = 0$, the second is checked analogously to (4.8).

4.4 Comparison of different gauges

Let us compare semiclassical electrodynamics in different gauges (see table 1).

An important specific feature of gauge theories is that the operator 1-form $\Omega_X[\delta X]$ may be zero for some δX not of the trivial form $\delta S \neq 0$, $\delta\Pi = 0$, $\delta\Phi = 0$. All "zero modes" of the 1-form are presented in table 1. Check of properties of zero operators $\Omega_X[\delta X]$ for the Lorentz gauge case is analogous to the Hamiltonian gauge case.

Analogously to the scalar theories, one can check that any operator commuting with all $\Omega_X[\delta X]$ is a multiplicator by a c-number constant. For the Couloumb gauge, with no gauge freedom, it is evident; other gauges are equivalent to it.

5 Semiclassical observables and transformations

Field operators are important objects of quantum field theory. However, for gauge theories fields $\hat{A}^\mu(x)$, $\hat{\theta}(x)$ are not physical observables. Therefore, the more complicated gauge-invariant combinations of fields should be viewed as observables.

In classical mechanics, observables may be introduced in different ways. First, one can say that states of a classical system are specified by points of the phase space and an observable is given if its value in any state is specified. Observables then are viewed as real functions on the phase space.

Alternatively, any observable may be also viewed as a classical Hamiltonian generating an evolution transformation group. Thus, one can say that an observable is specified if a one-parametric group of symplectic transformation is given.

Analogously, quantum observables may be specified by Hermitian operators, as well as by unitary evolution groups.

Table 1: Semiclassical bundle for different gauges

Base $\mathcal{X}$ of the semiclassical bundle	**Hamiltonian gauge:** Set of all $(S, \mathcal{E}_k(\mathbf{x}), \mathcal{A}^k(\mathbf{x}), \Pi_\theta(\mathbf{x}), \Theta(\mathbf{x}))$ such that $\partial_k \mathcal{E}_k + i(\Pi^*_\theta \Theta - \Pi_\theta \Theta^*) = 0$ **Couloumb gauge:** Set of all $(S, \mathcal{E}^\perp_k(\mathbf{x}), \mathcal{A}^k_\perp(\mathbf{x}))$ **Lorentz gauge:** Set of all $(S, \mathcal{E}_0(\mathbf{x}), \mathcal{E}_k(\mathbf{x}), \mathcal{A}^0(\mathbf{x}), \mathcal{A}^k(\mathbf{x}), \Pi_\theta(\mathbf{x}), \Theta(\mathbf{x}))$ such that $\mathcal{E}_0 = 0$ and $\partial_k \mathcal{E}_k + i(\Pi^*_\theta \Theta - \Pi_\theta \Theta^*) = 0$
A fibre $\mathcal{F}_X$, $X \in \mathcal{X}$	**Hamiltonian gauge:** Space of functionals $g[a^k, \vartheta, \vartheta^*]$ with inner product $\int D a^k D\vartheta D\vartheta^* g^* \prod_{\mathbf{x}} \delta(\Xi \Lambda_{\mathbf{x}}) g$. The space should be factorized and completed. **Couloumb gauge:** Space of functionals $f[a^k_\perp, \vartheta, \vartheta^*]$ with inner product $\int D a^k_\perp D\vartheta D\vartheta^* f^* f$. **Lorentz gauge:** Space of functionals $v[a^0, a^k, \vartheta, \vartheta^*]$ with inner product $\int D\lambda D a^k D\vartheta D\vartheta^* (v(-i\lambda, a, \vartheta, \vartheta^*))^* v(i\lambda, a, \vartheta, \vartheta^*)$. An additional condition $\left(\frac{1}{i}\frac{\delta}{\delta a^0(\mathbf{x})} - \frac{i}{\sqrt{-\Delta}} \Xi \Lambda_{\mathbf{x}}\right) v(a^0, a^k, \vartheta, \vartheta^*) = 0$ is imposed; states $v \sim v + \int d\mathbf{x} \left(\frac{1}{i}\frac{\delta}{\delta a^0(\mathbf{x})} + \frac{i}{\sqrt{-\Delta}} \Xi \Lambda_{\mathbf{x}}\right) \zeta$ are set to be equivalent.
Correspondence between gauges	**Hamiltonian and Couloumb gauges:** $f = \prod_{\mathbf{x}} \delta(\Xi \Lambda_{\mathbf{x}}) g$ **Hamiltonian and Lorentz gauges:** $v(a^0, a^k, \vartheta, \vartheta^*) = e^{-\int d\mathbf{x} a^0(\mathbf{x}) \frac{1}{\sqrt{-\Delta}} \Xi \Lambda_{\mathbf{x}}} g[a^k, \vartheta, \vartheta^*]$
1-form ω	**Hamiltonian gauge:** $\omega_X[\delta X] = \int d\mathbf{x} [\mathcal{E}_k \delta \mathcal{A}^k + \Pi_\theta \delta \Theta^* + \Pi^*_\theta \delta \Theta] - \delta S$ **Couloumb gauge:** $\omega_X[\delta X] = \int d\mathbf{x} [\mathcal{E}^\perp_k \delta \mathcal{A}^k_\perp + \Pi_\theta \delta \Theta^* + \Pi^*_\theta \delta \Theta] - \delta S$ **Lorentz gauge:** $\omega_X[\delta X] = \int d\mathbf{x} [\mathcal{E}_k \delta \mathcal{A}^k + \Pi_\theta \delta \Theta^* + \Pi^*_\theta \delta \Theta] - \delta S$
1-form Ω	**Hamiltonian gauge:** $\Omega_X[\delta X] = \int d\mathbf{x} [\delta \mathcal{E}_k a^k + \delta \Pi_\theta \vartheta^* + \delta \Pi^*_\theta \vartheta - \delta \mathcal{A}^k \frac{1}{i}\frac{\delta}{\delta a^k} - \delta \Theta \frac{1}{i}\frac{\delta}{\delta \vartheta} - \delta \Theta^* \frac{1}{i}\frac{\delta}{\delta \vartheta^*}]$ **Couloumb gauge:** $\Omega_X[\delta X] = \int d\mathbf{x} [\delta \mathcal{E}^\perp_k a^k_\perp + \delta \Pi_\theta \vartheta^* + \delta \Pi^*_\theta \vartheta - \delta \mathcal{A}^k_\perp \frac{1}{i}\frac{\delta}{\delta a^k_\perp} - \delta \Theta \frac{1}{i}\frac{\delta}{\delta \vartheta} - \delta \Theta^* \frac{1}{i}\frac{\delta}{\delta \vartheta^*}]$ **Lorentz gauge:** $\Omega_X[\delta X] = \int d\mathbf{x} [\delta \mathcal{E}_k a^k + \delta \Pi_\theta \vartheta^* + \delta \Pi^*_\theta \vartheta - \delta \mathcal{A}^0 \frac{1}{i}\frac{\delta}{\delta a^0} - \delta \mathcal{A}^k \frac{1}{i}\frac{\delta}{\delta a^k} - \delta \Theta \frac{1}{i}\frac{\delta}{\delta \vartheta} - \delta \Theta^* \frac{1}{i}\frac{\delta}{\delta \vartheta^*}]$
Zero operators $\Omega_X[\delta X] \sim 0$	**all gauges:** $\delta X = (\delta S, \delta \Pi = 0, \delta \Phi = 0)$ **Hamiltonian and Lorentz gauges:** $\delta X = \nabla_{\Lambda[\beta]} X$; this means that $\nabla_{\Lambda[\beta]} \mathcal{E}_k = 0$, $\nabla_{\Lambda[\beta]} \Pi_\theta = i\beta \Pi_\theta$, $\nabla_{\Lambda[\beta]} \Pi^*_\theta = -i\beta \Pi^*_\theta$, $\nabla_{\Lambda[\beta]} \mathcal{A}^k = -\partial_k \beta$; $\nabla_{\Lambda[\beta]} \Theta = i\beta \Theta$, $\nabla_{\Lambda[\beta]} \Theta^* = -i\beta \Theta^*$ **Lorentz gauge:** $\delta X = \nabla_{\mathcal{E}_0[\kappa]} X$: this means that $\nabla_{\mathcal{E}_0[\kappa]} \mathcal{A}^0 = \beta$, other variations are zero.

These conceptions may be considered in the semiclassical theory as well. Subsection 5.1 deals with semiclassical investigation of observables viewed as Hermitian operators. Subsection 5.2 is devoted to semiclassical analysis of evolution generated by semiclassical observables. Examples are gauge and Poincare transformations (subsections 5.3 and 5.4).

Heisenberg fields are very important objects of quantum field theory. Their semiclassical analogs are investigated in subsection 5.5.

Gauge equivalence relation should conserve under Poincare transformations: gauge equivalent semiclassical states should be taken to gauge equivalent. This property is discussed in subsection 5.6.

To check Poincare group relation (2.32), it is convenient to reduce it to its infinitesimal Lie algebra analog. This problem is considered in subsection 5.7.

5.1 Semiclassical observables

First of all consider the observables in the Hamiltonian approach. Suppose them to depend on the small parameter $\sqrt{h}$, fields $\hat{\varphi} = (\hat{A}^\mu, \hat{\theta}, \hat{\theta}^*)$ and momenta $\hat{\pi} = (\hat{E}_\mu, \hat{\pi}_\theta, \hat{\pi}_\theta^*)$ as

$$\hat{O}^h = O(\sqrt{h}\hat{\varphi}, \sqrt{h}\hat{\pi}). \tag{5.1}$$

Operators (5.1) will be called semiclassical. It is supposed in qunatum field theory that expression (5.1) is well-defined iff $O(\Phi, \Pi)$ is a gauge-invariant functional: it should not change under transformation

$$\Theta \to \Theta e^{-i\alpha}, \quad \Pi_\theta \to \Pi_\theta e^{-i\alpha}, \quad \mathcal{A}^k \to \mathcal{A}^k + \partial_k \alpha, \quad \mathcal{A}^0 \to \mathcal{A}^0 + \kappa.$$

Apply the semiclassical operator (5.1) to the semiclassical state $K_X^h f$. The general structure of the commutation rule is aas follows

$$\hat{O}^h K_X^h f = K_X^h (O(X) + \sqrt{h}(\Xi O)(X) + \frac{h}{2}(\Xi^2 O)(X) + ...)f. \tag{5.2}$$

For different gauges, explicit forms of the operators $\Xi^n O$ (for example, ΞO) are presented in table 2.

One can take the operator ΞO to the canonical form (2.39). The tangent vectors $\nabla_O X$ to the base of the semiclassical bundle are also calculated in table 2 for different gauges ($\nabla_O X$ appears to be a tangent vector, provided that the gauge invariance conditions are satisfied). Since the operaotrs $\Omega_X[\nabla_O X]$ are well-defined according to the previous section, the operator $\Xi O(X)$ also takes zero-norm states to zero-norm states and conserves the linearized Gupta-Bleuler condition (4.22).

Table 2: Semiclassical observables for different gauges

Operators $\Xi^n O(X)$	**Hamiltonian gauge:** $$(\Xi^n_H O)(X) = \frac{\partial^n}{\partial\sqrt{h}^n}\|_{h=0} O(\mathcal{A}^k + \sqrt{h}a^k, \mathcal{E}_k + \frac{\sqrt{h}}{i}\frac{\delta}{\delta a^k}, \Theta + \sqrt{h}\vartheta, \Theta^* + \sqrt{h}\vartheta^*, \Pi_\theta + \frac{\sqrt{h}}{i}\frac{\delta}{\delta\vartheta^*}, \Pi^*_\theta + \frac{\sqrt{h}}{i}\frac{\delta}{\delta\vartheta})$$ **Couloumb gauge:** $$(\Xi^n_C O)(X) = \frac{\partial^n}{\partial\sqrt{h}^n}\|_{h=0} O(\mathcal{A}^k_\perp + \sqrt{h}a^k_\perp, \mathcal{E}_k + \sqrt{h}\hat{\varepsilon}_k - h\frac{i}{\partial^2}\partial_k(\vartheta\frac{1}{i}\frac{\delta}{\delta\vartheta} - \varphi^*\frac{1}{i}\frac{\delta}{\delta\vartheta^*}), \Theta + \sqrt{h}\vartheta, \Theta^* + \sqrt{h}\vartheta^*, \Pi_\theta + \frac{\sqrt{h}}{i}\frac{\delta}{\delta\vartheta^*}, \Pi^*_\theta + \frac{\sqrt{h}}{i}\frac{\delta}{\delta\vartheta}),$$ $$\mathcal{E}_k = (\delta_{kl} - \frac{\partial_k\partial_l}{\partial^2})\mathcal{E}^\perp_k - \frac{1}{\partial^2}\partial_k i(\Pi^*_\theta\Theta - \Pi_\theta\Theta^*);$$ $$\hat{\varepsilon}_k = (\delta_{kl} - \frac{\partial_k\partial_l}{\partial^2})\frac{1}{i}\frac{\delta}{\delta a^l_\perp} - \frac{1}{\partial^2}\partial_k(i\Pi^*_\theta\vartheta + \Theta\frac{\delta}{\delta\vartheta} - i\Pi_\theta\vartheta^* - \Theta^*\frac{\delta}{\delta\vartheta^*})$$ **Lorentz gauge:** $$(\Xi^n_L O)(X) = \frac{\partial^n}{\partial\sqrt{h}^n}\|_{h=0} O(\mathcal{A}^k + \sqrt{h}a^k, \mathcal{E}_k + \frac{\sqrt{h}}{i}\frac{\delta}{\delta a^k}, \mathcal{A}^0 + \sqrt{h}a^0, \frac{\sqrt{h}}{i}\frac{\delta}{\delta a^0}, \Theta + \sqrt{h}\vartheta, \Theta^* + \sqrt{h}\vartheta^*, \Pi_\theta + \frac{\sqrt{h}}{i}\frac{\delta}{\delta\vartheta^*}, \Pi^*_\theta + \frac{\sqrt{h}}{i}\frac{\delta}{\delta\vartheta}).$$
Partial case: $\Xi O(X)$	**Hamiltonian gauge:** $\Xi_H O = \int d\mathbf{x}(\frac{\delta O}{\delta\mathcal{A}^k(\mathbf{x})}a^k(\mathbf{x}) + \frac{\delta O}{\delta\mathcal{E}_k(\mathbf{x})}\frac{1}{i}\frac{\delta}{\delta a^k(\mathbf{x})} + \frac{\delta O}{\delta\Theta(\mathbf{x})}\vartheta(\mathbf{x}) + \frac{\delta O}{\delta\Theta^*(\mathbf{x})}\vartheta^*(\mathbf{x}) + \frac{\delta O}{\delta\Pi_\theta(\mathbf{x})}\frac{1}{i}\frac{\delta}{\delta\vartheta^*(\mathbf{x})} + \frac{\delta O}{\delta\Pi^*_\theta(\mathbf{x})}\frac{1}{i}\frac{\delta}{\delta\vartheta(\mathbf{x})})$ **Couloumb gauge:** $\Xi_C O = \Xi_H O - \int d\mathbf{x}\left(\frac{\delta O}{\delta\mathcal{A}^k(\mathbf{x})}\frac{1}{\partial^2}\partial_k\Xi_H\partial_l\mathcal{A}^l(\mathbf{x}) + \frac{\delta O}{\delta\mathcal{E}_k(\mathbf{x})}\frac{1}{\partial^2}\partial_k\Xi_H\Lambda_{\mathbf{x}}\right)$ **Lorentz gauge:** $\Xi_L O = \Xi_H O + \int d\mathbf{x}\left(\frac{\delta O}{\delta\mathcal{A}^0(\mathbf{x})}a^0(\mathbf{x}) + \frac{\delta O}{\delta\mathcal{E}_0(\mathbf{x})}\frac{1}{i}\frac{\delta}{\delta a^0(\mathbf{x})}\right)$
$\Xi O(X) = -\Omega[\nabla_O X]$; form of $\nabla_O X$	**Hamiltonian gauge:** $\nabla_O\mathcal{A}^k = \frac{\delta O}{\delta\mathcal{E}_k}$; $\nabla_O\mathcal{E}_k = -\frac{\delta O}{\delta\mathcal{A}^k}$; $\nabla_O\Theta = \frac{\delta O}{\delta\Pi^*_\theta}$; $\nabla_O\Pi_\theta = -\frac{\delta O}{\delta\Theta^*}$; $\nabla_O\Theta^* = \frac{\delta O}{\delta\Pi_\theta}$; $\nabla_O\Pi^*_\theta = -\frac{\delta O}{\delta\Theta}$. **Couloumb gauge:** $\nabla_O\mathcal{A}^k = \frac{\delta O}{\delta\mathcal{E}_k} - \partial_k\frac{1}{\partial^2}\partial_l\frac{\delta O}{\delta\mathcal{E}_l}$; $\nabla_O\mathcal{E}^\perp_k = -\frac{\delta O}{\delta\mathcal{A}^k}$; $\nabla_O\Theta = \frac{\delta O}{\delta\Pi^*_\theta} + i\Theta\frac{1}{\partial^2}\partial_l\frac{\delta O}{\delta\mathcal{E}_l}$; $\nabla_O\Pi_\theta = -\frac{\delta O}{\delta\Theta^*} + i\Pi_\theta\frac{1}{\partial^2}\partial_l\frac{\delta O}{\delta\mathcal{E}_l}$; $\nabla_O\Theta^* = \frac{\delta O}{\delta\Pi_\theta} - i\Theta^*\frac{1}{\partial^2}\partial_l\frac{\delta O}{\delta\mathcal{E}_l}$; $\nabla_O\Pi^*_\theta = -\frac{\delta O}{\delta\Theta} - i\Pi^*_\theta\frac{1}{\partial^2}\partial_l\frac{\delta O}{\delta\mathcal{E}_l}$. **Lorentz gauge:** $\nabla_O\mathcal{A}^k = \frac{\delta O}{\delta\mathcal{E}_k}$; $\nabla_O\mathcal{E}_k = -\frac{\delta O}{\delta\mathcal{A}^k}$; $\nabla_O\mathcal{A}^0 = \frac{\delta O}{\delta\mathcal{E}_0}$; $\nabla_O\mathcal{E}_0 = -\frac{\delta O}{\delta\mathcal{A}^0}$; $\nabla_O\Theta = \frac{\delta O}{\delta\Pi^*_\theta}$; $\nabla_O\Pi_\theta = -\frac{\delta O}{\delta\Theta^*}$; $\nabla_O\Theta^* = \frac{\delta O}{\delta\Pi_\theta}$; $\nabla_O\Pi^*_\theta = -\frac{\delta O}{\delta\Theta}$.
Gauge invariance condition	**Hamiltonian and Lorentz gauges:** $d\Lambda[\beta][\nabla_O X] = 0$ **Lorentz gauge:** $d\mathcal{E}_0[\beta][\nabla_O X] = 0$

5.2 Transformations of semiclassical states

1. Poincare and gauge transformations are of the form

$$\exp\left[-\frac{i}{h}\tau\hat{O}\right], \tag{5.3}$$

where $\hat{O}$ is of the semiclassical form (5.1). Due to renormalization, one should also take into account the one-loop quantum corrections and consider observables of the more general form

$$\hat{O} = O(\sqrt{h}\hat{\varphi}, \sqrt{h}\hat{\pi}) + hO^{(1)}(\sqrt{h}\hat{\varphi}, \sqrt{h}\hat{\pi}) + ...$$

Let us consider the state vector

$$\Psi^\tau \equiv e^{-\frac{i}{h}\hat{O}\tau} K^h_{X_0} f_0$$

as $h \to 0$. It satisfies the following Cauchy problem

$$i\frac{d\Psi^\tau}{d\tau} = \frac{1}{h}\hat{O}\Psi^\tau, \quad \Psi^0 = K^h_{X_0} f_0. \tag{5.4}$$

Let us look for the approximate as $h \to 0$ solutions of the Cauchy problem (5.4) in a following form:

$$\Psi^\tau \simeq K^h_{X^\tau} f^\tau. \tag{5.5}$$

Let us find semiclassical equations for X^τ, f^τ. It happens that commutation rule (2.13) is to be corrected for our case as:

$$ih\frac{d}{d\tau}K^h_{X^\tau} f^\tau = K^h_{X^\tau}[\omega_{X^\tau}[\dot{X}^\tau]] - \sqrt{h}\Omega_{X^\tau}[\dot{X}^\tau] + ih\frac{d}{d\tau}]f^\tau; \tag{5.6}$$

no additional terms of the order $O(h)$ are added. Combining commutation rules (5.5) and (5.6), one finds that substitution (5.5) is an approximate solution of eq.(5.4) iff

$$\omega_{X^\tau}[\dot{X}^\tau] = O(X^\tau); \quad \Omega_{X^\tau}[\dot{X}^\tau] = -\Xi O(X^\tau); \tag{5.7}$$

$$i\frac{d}{d\tau}f^\tau = \left[\frac{1}{2}\Xi^2 O((X^\tau) + O^{(1)}(X^\tau)\right] f^\tau. \tag{5.8}$$

Eqs. (5.7) specify classical evolution. The first relation allows us to express $\dot{S}^\tau$ via other derivatives. The second equations can be written as

$$\dot{X}^\tau = \nabla_O X^\tau + \overline{\delta}X, \quad \Omega_X[\overline{\delta}X] = 0.$$

We see that $\dot{X}^\tau$ is defined up to a gauge transformation. One can choose $\overline{\delta}X$ in order to make equations more convenient.

Denote by $u_O^\tau : X \mapsto u_O^\tau X$ the transformation taking the initial conditions for the system of equations for $X^\tau \equiv (S^\tau, \Pi^\tau, \Phi^\tau)$ of the form

$$\dot{S}^\tau = \int d\mathbf{x}\Pi^\tau\dot{\Phi}^\tau - O(\Pi^\tau, \Phi^\tau), \quad \dot{\Pi}^\tau = \nabla_O\Pi^\tau, \quad \dot{\Phi}^\tau = \nabla_O\Phi^\tau \tag{5.9}$$

to the solution of the Cauchy problem. This is the classical evolution corresponding to the observable O.

2. Investigate the properties of evolution of f^t which is given by eq.(5.8). Let $X^\tau(\alpha)$ be a function of τ and $\alpha = (\alpha_1, ..., \alpha_k)$. It happens that the following relation is satisfied:

$$\begin{array}{c} [i\frac{d}{d\tau} - \frac{1}{2}\Xi^2 O(X^\tau) - O^{(1)}(X^\tau); \Omega_{X^\tau}[\frac{\partial X^\tau}{\partial\alpha_a}]] = \\ i\Omega_{X^\tau}[\frac{\partial}{\partial\alpha_a}(\frac{\partial X^\tau}{\partial\tau} - \nabla_O X^\tau)] \end{array} \tag{5.10}$$

One can check equality (5.10) in differeny ways. First, one can start from the identity

$$[ih\frac{d}{d\tau} - \hat{O}^h, ih\frac{\partial}{\partial\alpha_a}]K^h_{X^\tau(\alpha)}f^\tau(\alpha) = 0.$$

It is taken to the following form

$$\begin{array}{c} [\omega_X[\dot{X}] - \sqrt{h}\Omega_X[\dot{X}] + ih\frac{\partial}{\partial\tau} - O(X) - \sqrt{h}\Xi O(X) - \frac{h}{2}\Xi^2 O(X) - O^{(1)}(X); \\ \omega_X[\frac{\partial X}{\partial\alpha_a} - \sqrt{h}\Omega_X[\frac{\partial X}{\partial\alpha_a}] + ih\frac{\partial}{\partial\alpha_a}] = 0. \end{array}$$

Considering the terms of the order $O(h^{3/2})$, one comes to the identity (5.10). Another way to check eq.(5.10) is to use the direct calculation method. It is important to notice that the operator identity (5.10) is valid even for the space of functionals $g[a^k, \vartheta, \vartheta^*]$ before factorization.

3. Let us investigate the unitariry proprty for the Hamiltonian gauge. It happens that one should require that

$$[\nabla_O; \nabla_{\Lambda[\beta]}] = -\nabla_{\Lambda[C_O\beta]} \tag{5.11}$$

for some linear operator C_O. Under condition (5.11), let us check that equality

$$\begin{array}{c} [i\frac{d}{d\tau} - \frac{1}{2}\Xi^2 O(X^\tau) - O^{(1)}(X^\tau); (\Xi\Lambda[\beta])(X^\tau)] = \\ i(\Xi\Lambda[C_O\beta])(X^\tau)] \end{array} \tag{5.12}$$

is valid for the space of functionals $g[a^k,\vartheta,\vartheta^*]$ before factorization.

To justify property (5.12), notice that it is equivalent to

$$\begin{gathered}[i\tfrac{d}{d\tau}-\tfrac{1}{2}\Xi^2 O(X^\tau)-O^{(1)}(X^\tau);\Omega_{X^\tau}[\nabla_{\Lambda[\beta]}X^\tau]= \\ i\Omega_{X^\tau}[\nabla_{\Lambda[C_O\beta]}X^\tau].\end{gathered} \tag{5.13}$$

To check relation (5.13), leu us use identity (5.10). Set

$$X^\tau(\alpha)\equiv u^\alpha_{\Lambda[\beta]}u^\tau_O X = u^\alpha_{\Lambda[\beta]}X^\tau.$$

Then $\nabla_{\Lambda[\beta]}X^\tau \equiv \frac{\partial X^\tau(\alpha)}{\partial\alpha}$ and property (5.13) is taken to

$$\begin{gathered}\nabla_{\Lambda[C_O\beta]}X^\tau = \tfrac{\partial}{\partial\alpha}\Big|_{\alpha=0}\left(\tfrac{\partial X^\tau(\alpha)}{\partial\tau}-\nabla_O X^\tau(\alpha)\right)= \\ \tfrac{\partial}{\partial\alpha}\Big|_{\alpha=0}\tfrac{\partial}{\partial t}\Big|_{t=0}\left(u^\alpha_{\Lambda[\beta]}u^{t+\tau}_O - u^t_O u^\alpha_{\Lambda[\beta]}u^\tau_O\right)X\ =[\nabla_{\Lambda[\beta]};\nabla_O]X^\tau.\end{gathered}$$

Thus, relation (5.12) is satisfied.

Let us now check conservation of the inner product

$$(g^\tau,\prod_{\mathbf{x}}\delta(\Xi\Lambda_{\mathbf{x}})(X^0)g^\tau)=(g^\tau,\int D\beta e^{i(\Xi\Lambda[\beta])(X^\tau)}g^\tau)$$

where

$$(g,\tilde g)\equiv Da^k D\vartheta D\vartheta^* g^*[a^k,\vartheta,\vartheta^*]\tilde g[a^k,\vartheta,\vartheta^*]$$

It follows from eq.(5.12) that

$$\begin{gathered}[i\tfrac{d}{d\tau}-\tfrac12\Xi^2 O(X^\tau)-O^{(1)}(X^\tau);e^{i\Xi\Lambda[\beta](X^\tau)}] \\ =-\Xi\Lambda[C_O\beta](X^\tau)e^{i\Xi\Lambda[\beta](X^\tau)}=-\textstyle\int d\mathbf{y}(C_O\beta)(\mathbf{y})\frac1i\frac{\delta}{\delta\beta(\mathbf{y})}e^{i\Xi\Lambda[\beta](X^\tau)}].\end{gathered}$$

Therefore,

$$i\frac{d}{d\tau}(g^\tau,\prod_{\mathbf{x}}\delta(\Xi\Lambda_{\mathbf{x}})(X^0)g^\tau)=[O^{(1)}(X^\tau)-O^{(1)*}(X^\tau)-iTrC_O(X^\tau)](g^\tau,\prod_{\mathbf{x}}\delta(\Xi\Lambda_{\mathbf{x}})(X^0)g^\tau).$$

Thus, zero-norm states are always taken to zero-norm states under condition (5.12), while unitary requirements mean that

$$ImO^{(1)}=\frac12 TrC_O. \tag{5.14}$$

Usually, TrC_O will vanish.

4. For the Lorentz gauge, one should check conservation of the linearized Gupta-Bleuler condition. A sufficient condition is as follows:

$$\begin{gathered}[i\frac{d}{d\tau} - \frac{1}{2}\Xi^2 O(X^\tau) - O^{(1)}(X^\tau); \Xi\mathcal{E}_0[\beta](X^\tau) - i\Xi\Lambda[\frac{1}{\sqrt{-\Delta}}\beta](X^\tau)] = \\ i(\Xi\mathcal{E}_0[C_O^{(L)}\beta](X^\tau) - i\Xi\Lambda[\frac{1}{\sqrt{-\Delta}}C_O^{(L)}\beta](X^\tau)).\end{gathered} \tag{5.15}$$

for some operator $C_O^{(L)}$. It is a corollary of the relation

$$[\nabla_O; \nabla_{\mathcal{E}_O[\beta]-i\Lambda[\frac{1}{\sqrt{-\Delta}}\beta]}] = \nabla_{\mathcal{E}_O[C_O^{(L)}\beta]-i\Lambda[\frac{1}{\sqrt{-\Delta}}C_O^{(L)}\beta]}. \tag{5.16}$$

5. Thus, for all observables we have constructed the semiclassical evolution transformation taking initial condition for eq.(5.8) to the solution for this equation. This transformation conserves equivalence property and inner product. It can be reduced to the factorspace; denote the obtained operator as $U_O^\tau(u_O^\tau X \leftarrow X) : \mathcal{F}_X \to \mathcal{F}_{u_O^\tau X}$.

The introduced transformations u_O^τ and $U_O^\tau(u_O^\tau X \leftarrow X)$ obey the following properties. Let $X = X(\alpha)$; then

$$\begin{gathered}\omega_{u_O X}[\frac{\partial(u_O X)}{\partial\alpha_a}] = \omega_X[\frac{\partial X}{\partial\alpha_a}]; \\ \Omega_{u_O X}[\frac{\partial(u_O X)}{\partial\alpha_a}]U_O(u_O X \leftarrow X) = U_O(u_O X \leftarrow X)\Omega_X[\frac{\partial X}{\partial\alpha_a}].\end{gathered} \tag{5.17}$$

The first relation means that the action 1-form ω is conserved under time evolution

$$\frac{d}{d\tau}\omega_{u_O^\tau X}[\frac{\partial(u_O^\tau X)}{\partial\alpha_a}] = 0. \tag{5.18}$$

Relation (5.18) is checked by a direct computation. The second property means that the operator $\Omega_{u_O^\tau X}[\frac{\partial(u_O^\tau X)}{\partial\alpha_a}]$ takes solutions of eq.(5.8) to solutions. This is true since $\Omega_{u_O^\tau X}[\frac{\partial(u_O^\tau X)}{\partial\alpha_a}]$ commutes with $i\frac{d}{d\tau} - \frac{1}{2}\Xi^2 O(X^\tau) - O^{(1)}(X^\tau)$ according to eq.(5.10).

5.3 Semiclassical gauge transformations

1. It has been noticed in section 3 that quantum states

$$\Psi_H \sim \exp\left[-\frac{i}{\sqrt{h}}\int d\mathbf{x}\alpha(\mathbf{x})\hat{\Lambda}_{\mathbf{x}}\right]\Psi_H \tag{5.19}$$

are gauge-equivalent (for the Hamiltonian gauge). Therefore, semiclassical states

$$K^h_{X_1} f_1 \sim e^{-\frac{i}{\sqrt{h}}\int d\mathbf{x}\alpha(\mathbf{x})\hat{\Lambda}_{\mathbf{x}}} K^h_{X_1} f_1 \simeq K^h_{u_{\Lambda[\alpha]}X} U_{\Lambda[\alpha]}(u_{\Lambda[\alpha]}X \leftarrow X) f$$

are also gauge-equivalent. Therefore, for the Hamiltonian gauge, one should introduce an equivalence relation on the semiclassical bundle: one should set

$$X_1 \sim X_2 \Leftrightarrow X_2 = u_{\Lambda[\alpha]} X_1 \text{ for some } \alpha;$$

moreover, $K^h_{X_1} f_1 \simeq K^h_{X_2} f_2$ iff $f_2 = V(X_2 \leftarrow X_1) f_1$ for

$$V(X_2 \leftarrow X_1) = U_{\Lambda[\alpha]}(u_{\Lambda[\alpha]} X \leftarrow X).$$

2. For the Lorentz gauge, due the Gupta-Bleuler equivalence relation, there is also a gauge transformation of another form

$$\Psi_L \sim e^{-\frac{i}{\sqrt{h}} \int d\mathbf{x} \kappa(\mathbf{x}) \hat{E}_0(\mathbf{x})} \Psi_L.$$

For the Lorentz gauge, one should then set

$$X_1 \sim X_2 \Leftrightarrow X_2 = u_{\Lambda[\alpha]} u_{\mathcal{E}_0[\kappa]} X_1 \text{ for some } \alpha, \kappa;$$

$K^h_{X_1} f_1 \simeq K^h_{X_2} f_2$ iff $f_2 = V(X_2 \leftarrow X_1) f_1$ for

$$V(X_2 \leftarrow X_1) = U_{\Lambda[\alpha]}(u_{\Lambda[\alpha]} u_{\mathcal{E}_0[\kappa]} X \leftarrow u_{\mathcal{E}_0[\kappa]} X) U_{\mathcal{E}_0[\kappa]}(u_{\mathcal{E}_0[\kappa]} X \leftarrow X).$$

3. An explicit form of equivalence relation is the following. For the Hamiltonian gauge, property $X_2 = u_{\Lambda[\alpha]} X_1$ means that

$$S^{(2)} = S^{(1)}, \quad \mathcal{E}^{(2)}_k = \mathcal{E}^{(1)}_k, \quad \mathcal{A}^{(2)}_k = \mathcal{A}^{(1)}_k - \partial_k \alpha, \quad \Pi^{(2)} = \Pi^{(1)} e^{i\alpha}, \quad \Theta^{(2)} = \Theta^{(1)} e^{i\alpha}. \quad (5.20)$$

The operator $V(X_2 \leftarrow X_1)$ is if the form

$$V(X_2 \leftarrow X_1) g[a^k, \vartheta, \vartheta^*] = g[a^k, \vartheta e^{-i\alpha}, \vartheta^* e^{i\alpha}]. \quad (5.21)$$

For the Lorentz gauge, equality $X_2 = u_{\Lambda[\alpha]} u_{\mathcal{E}_0[\kappa]} X_1$ consists of relation $\mathcal{A}^{(2)}_0 = \mathcal{A}^{(1)}_0 + \kappa$ and eqs.(5.20), the operator $V(X_2 \leftarrow X_1)$ is

$$V(X_2 \leftarrow X_1) v[a^0, a^k, \vartheta, \vartheta^*] = V(X_2 \leftarrow X_1) v[a^0, a^k, \vartheta e^{-i\alpha}, \vartheta^* e^{i\alpha}]. \quad (5.22)$$

It follows from relations (5.21), (5.22) that

$$V(X_3 \leftarrow X_1) = V(X_3 \leftarrow X_2) V(X_2 \leftarrow X_1) \quad (5.23)$$

for both gauges.

Notice also that properies (2.31) are partial cases of (5.17).

5.4 Semiclassical Poincare transformations

To construct semiclassical Poincare transformations (classical transformations $u_g : \mathcal{X} \to \mathcal{X}$ and unitary operators $U_g(u_g X \leftarrow X) : \mathcal{F}_X \to \mathcal{F}_{u_g X}$, notice that it is possible to use decomposition (3.22) for the Poincare group. Thus, it is sufficient to specify semiclassical spatial translations and rotations, boosts and evolution.

The corresponding 1-parametric subgroups $g(\tau) \equiv (a_\tau, \Lambda_\tau)$ of the Poincare group are presented in table 3. For such cases,

$$u_{g(\tau)} \equiv u_O^\tau, \quad U_{g(\tau)}(u_{g(\tau)} X \leftarrow X) \equiv U_O^\tau(u_O^\tau X \leftarrow X).$$

Observables O corresponding to 1-parametric subgroups are presented in table 3. They indeed satidfy eq.(5.11).

Notice that properties (2.21) of semiclassical Poincare transformations are partial cases of (5.17).

5.5 Manifestly covariant semiclassical observables and fields

1. In the previous subsections, we have considered the semiclassical field operators in the Hamiltonian framework. The fields depended on the spatial coordinates only.

Let us consider now the Poincare covariant observables. They should depend on Heisenberg fields $\hat{\varphi}(x) = (\hat{A}^\mu(x), \hat{\theta}(x), \hat{\theta}^*(x))$:

$$\hat{O} = O(\sqrt{h}\hat{\varphi}(\cdot)) = O(\sqrt{h}\hat{A}^\mu(\cdot), \sqrt{h}\hat{\theta}(\cdot), \sqrt{h}\hat{\theta}^*(\cdot)). \tag{5.24}$$

For gauge theories, only gauge-invariant observables should be considered. This means that quantum expression (5.24) specidies an observable iff the classical functional $O(\mathcal{A}^\mu(\cdot), \Theta(\cdot), \Theta^*(\cdot))$ is invariant under gauge transformations

$$O(\mathcal{A}^\mu + \partial_\mu \alpha, \Theta e^{i\alpha}, \Theta^* e^{-i\alpha}) = O(\mathcal{A}^\mu, \Theta, \Theta^*). \tag{5.25}$$

One can rewrite property (5.25) in the infinitesimal form. Namely, expanding the left-hand side of relation (5.25) in $\alpha(\cdot)$, one finds that

$$\int dx \left[\frac{\delta O}{\delta \mathcal{A}^\mu(x)} \partial_\mu \alpha(x) + \frac{\delta O}{\delta \Theta(x)} i\alpha(x)\Theta(x) + \frac{\delta O}{\delta \Theta^*(x)} (-i\alpha(x)\Theta(x)) \right] = 0.$$

This means that

$$\partial_\mu \frac{\delta O}{\delta \mathcal{A}^\mu(x)} = i \left[\Theta(x) \frac{\delta O}{\delta \Theta(x)} - \Theta^*(x) \frac{\delta O}{\delta \Theta^*(x)} \right]. \tag{5.26}$$

Table 3: Equations for classical Poincare transformations

Element of Poincare group (a_τ, Λ_τ); corresponding observable $\hat{O}$	Classical Poincare transformation $u_{a_\tau,\Lambda_\tau} : X^0 \mapsto X^\tau$ is found from classical equations $\dot{\Phi} = \nabla_O \Phi$, $\dot{\Pi} = \nabla_O \Pi$, $\dot{S} = \int d\mathbf{x} \Pi \dot{\Phi} - O(\Pi, \Phi)$ of the form:
$a_\tau = 0$, $\Lambda_\tau = \exp(\frac{\tau}{2} l^{sm} \zeta_{sm})$; $\zeta_{sm} = -\zeta_{ms}$, spatial rotation; $\hat{O} = -\frac{1}{2}\mathcal{M}^{lm}\zeta_{lm}$.	**all gauges:** $\dot{\Theta}^\tau = \zeta_{kl} x^k \partial_l \Theta^\tau; \quad \dot{\Pi_\theta}^\tau = \zeta_{kl} x^k \partial_l \Pi^\tau_\theta; \quad \dot{S}^\tau = 0;$ $\dot{\mathcal{A}}^{s\tau} = \zeta_{kl} x^k \partial_l \mathcal{A}^{s\tau} + \zeta_{sl} \mathcal{A}^{l\tau}; \quad \dot{\mathcal{E}}^\tau_s = \zeta_{kl} x^k \partial_l \mathcal{E}^\tau_s + \zeta_{sl} \mathcal{E}^\tau_l.$ **Lorentz gauge:** $\dot{\mathcal{A}}^{0\tau} = \zeta_{kl} x^k \partial_l \mathcal{A}^{0\tau}$.
$a^0_\tau = 0$, $a^k_\tau = b^k \tau$, $\Lambda_\tau = 1$; spatial translation; $\hat{O} = b^k \mathcal{P}^k$.	**all gauges:** $\dot{\Theta}^\tau = -b^k \partial_k \Theta^\tau; \quad \dot{\Pi_\theta}^\tau = -b^k \partial_k \Pi^\tau_\theta; \quad \dot{S}^\tau = 0;$ $\dot{\mathcal{A}}^{s\tau} = -b^k \partial_k \mathcal{A}^{s\tau}; \quad \dot{\mathcal{E}}^\tau_s = -b^k \partial_k \mathcal{E}^\tau_s.$ **Lorentz gauge:** $\dot{\mathcal{A}}^{0\tau} = -b^k \partial_k \mathcal{A}^{0\tau}$.
$a^0_\tau = -\tau$, $a^k_\tau = 0$, $\Lambda_\tau = 1$; evolution; $\hat{O} = \mathcal{P}^0$.	**all gauges:** $\dot{\Theta}^\tau = \Pi^\tau_\theta + i\mathcal{A}^{0\tau}\Theta^\tau; \quad -\dot{\Pi_\theta}^\tau = -D_i D_i \Theta^\tau + m^2 \Theta^\tau + V'(\Theta^\tau \Theta^{\tau *})\Theta^\tau - i\mathcal{A}^{0\tau}\Pi^\tau_\theta;$ $\dot{\mathcal{A}}^{k\tau} = \mathcal{E}^\tau_k - \partial_k \mathcal{A}^{0\tau}; \quad -\dot{\mathcal{E}}^\tau_k = i[(D_k\Theta^\tau)^* \Theta^\tau - \Theta^{\tau *} D_k \Theta^\tau] - \partial_j(\partial_j \mathcal{A}^k - \partial_k \mathcal{A}^j);$ $\dot{S}^\tau = \int d\mathbf{x}[...];$ $D_k \equiv \partial_k + i\mathcal{A}^{k\tau}.$ **Hamiltonian gauge:** $\mathcal{A}^{0\tau} = 0$; **Couloumb gauge:** $\mathcal{A}^{0\tau} \Rightarrow \frac{1}{\partial^2}\partial_l \mathcal{E}^\tau_l$; **Lorentz gauge:** $\dot{\mathcal{A}}^{0\tau} = -\partial_k \mathcal{A}^{k\tau}$.
$a_\tau = 0$, $\Lambda_\tau = \exp(-\tau n^k l^{k0})$; boost; $\hat{O} = n^k \mathcal{M}^{k0}$.	**all gauges:** $\dot{\Theta}^\tau = n^s x^s [\Pi^\tau_\theta + i\mathcal{A}^{0\tau}\Theta^\tau]; \quad \dot{\mathcal{A}}^{k\tau} = x^s n^s \mathcal{E}^\tau_k - \partial_k(x^s n^s \mathcal{A}^{0\tau});$ $-\dot{\Pi_\theta}^\tau = -D_i x^s n^s D_i \Theta^\tau + x^s n^s(m^2\Theta^\tau + V'(\Theta^\tau\Theta^{\tau *})\Theta^\tau - i\mathcal{A}^{0\tau}\Pi^\tau_\theta);$ $-\dot{\mathcal{E}}^\tau_k = ix^s n^s[(D_k\Theta^\tau)^*\Theta^\tau - \Theta^{\tau *}D_k\Theta^\tau] - \partial_j x^s n^s(\partial_j \mathcal{A}^k - \partial_k \mathcal{A}^j);$ $\dot{S}^\tau = \int d\mathbf{x}[...];$ $D_k \equiv \partial_k + i\mathcal{A}^{k\tau}.$ **Hamiltonian gauge:** $\mathcal{A}^{0\tau} = 0$; **Couloumb gauge:** $x^s n^s \mathcal{A}^{0\tau} \Rightarrow \frac{1}{\partial^2}\partial_l x^s n^s \mathcal{E}^\tau_l$; **Lorentz gauge:** $\dot{\mathcal{A}}^{0\tau} = -\partial_k x^s n^s \mathcal{A}^{k\tau}$.

A formal analog of commutation rule (2.7) for the field is

$$\hat{O} K_X^h f \simeq K_X^h O(\mathcal{A}^\mu + \sqrt{h}\hat{a}^\mu, \Theta + \sqrt{h}\hat{\vartheta}, \Theta^* + \sqrt{h}\hat{\vartheta}^*) f \simeq K_X^h [O(X) + \sqrt{h}\Xi O(X)] f$$

with

$$\begin{gathered} O(X) \equiv O(\mathcal{A}^\mu(\cdot), \Theta(\cdot), \Theta^*(\cdot)), \\ \Xi O(X) \equiv \int dx \left(\frac{\delta O}{\delta \mathcal{A}^\mu(x)} \hat{a}^\mu(x|X) + \frac{\delta O}{\delta \Theta(x)} \hat{\vartheta}(x|X) + \frac{\delta O}{\delta \Theta^*(x)} \hat{\vartheta}^*(x|X) \right). \end{gathered} \tag{5.27}$$

Therefore, there is the following specific feature of the gauge theory. For the scalar field theory, semiclassical field $\hat{\phi}(x|X)$ is a well-defined operator distribution in the following sense: expression $\int dx \hat{\phi}(x|X) \frac{\delta O}{\delta \Phi(x)}$ specifies a well-defined operator for any smooth rapidly damping at infinity function $\frac{\delta O}{\delta \Phi(x)}$. For the electrodynamic case, the linear combination (5.4) should specify a well-defined operator, provided that the c-number functions $\frac{\delta O}{\delta \mathcal{A}^\mu(x)}$, $\frac{\delta O}{\delta \Theta(x)}$, $\frac{\delta O}{\delta \Theta^*(x)}$ satisfy eq.(5.26).

To define c-number quantity $O(X)$ and operator $\Xi O(X)$, let us introduce manifestly covariant notations.

2. Let us identify elements $X \in \mathcal{X}$ with sets $\overline{X} = (S, \overline{\Phi}(x)) \equiv (S, \overline{\mathcal{A}}^\mu(x), \overline{\Theta}(x), \overline{\Theta}^*(x)) \in \overline{\mathcal{X}} = \{(\overline{X})\}$ analogously to section 2. Here $\overline{\Phi}(x) \equiv \overline{\Phi}(x|X)$ is a solution of system of classical equations

$$\begin{gathered} \partial_\nu \overline{F}^{\mu\nu} = i(\overline{\Theta}^* \overline{D}^\mu \overline{\Theta} - \overline{\Theta} \overline{D}^\mu \overline{\Theta}^*); \\ \overline{D}_\mu \overline{D}^\mu \overline{\Theta} + m^2 \overline{\Theta} + V'(\overline{\Theta}^* \overline{\Theta}) \overline{\Theta} = 0 \end{gathered} \tag{5.28}$$

with

$$\overline{\mathcal{F}}^{\mu\nu} = \partial^\mu \overline{\mathcal{A}}^\nu - \partial^\nu \overline{\mathcal{A}}^\mu, \qquad \overline{D}_\mu = \partial_\mu - i\overline{\mathcal{A}}_\mu.$$

Initial conditions for system (5.28) are as follows:

$$\begin{gathered} \overline{\Theta}|_{x^0=0} = \Theta(\mathbf{x}), \quad \overline{D}_0 \overline{\Theta}|_{x^0=0} = \Pi_\theta(\mathbf{x}), \\ \overline{\mathcal{A}}^k|_{x^0=0} = \mathcal{A}^k(\mathbf{x}), \quad \overline{F}^{0k}|_{x^0=0} = \mathcal{E}_k(\mathbf{x}), \quad \overline{\mathcal{A}}^0|_{x^0=0} = \mathcal{A}^0(\mathbf{x}). \end{gathered} \tag{5.29}$$

Condition for $\overline{\mathcal{A}}^0$ should be imposed for Lorentz gauge only.

It is well-known that a solution to the Cauchy problem (5.28), (5.29) is defined up to a gauge transformation. Namely, if we constructed one of solutions $(\overline{\mathcal{A}}^\mu(x), \overline{\Theta}(x))$ then the functions

$$\overline{\mathcal{A}}^\mu(x) + \partial_\mu \rho(x), \qquad \overline{\Theta}(x) e^{i\rho(x)} \tag{5.30}$$

would also satisfy system (5.28). For different gauges, different additional gauge conditions are to be imposed then.

Making use of the introduced notations, set

$$O(X) \equiv O[\overline{\Phi}(\cdot)].$$

This is a well-defined expression since the functional O is invariant under gauge transformations (5.30).

For $X_1 \sim X_2$, one also checks property (2.41)

$$O(X_2) = O(X_1),$$

since gauge-equivalent initial conditions for system (5.28) generate gauge-equivalent solutions.

Let us check now property (2.36). It can be written as

$$O(\overline{\Phi}(\cdot|u_g X)) = (v_g O)(\overline{\Phi}(\cdot|X)) \equiv O(v_g \overline{\Phi}(\cdot|X)). \tag{5.31}$$

Property (5.31) means that the space-time functions

$$\tilde{\mathcal{A}}^\mu(x) = \Lambda^\mu_\nu \mathcal{A}^\nu(\Lambda^{-1}(x-a)); \qquad \tilde{\Theta}(x) = \Theta(\Lambda^{-1}(x-a)) \tag{5.32}$$

satisfies system (5.28), while initial conditions $\tilde{X} = (\tilde{\mathcal{E}}_\mu, \tilde{\mathcal{A}}^\mu, \tilde{\Pi}_\theta, \tilde{\Theta})$

$$\begin{array}{cc} \tilde{\Theta}(\mathbf{x}) \equiv \tilde{\Theta}|_{x^0=0}, & \tilde{\Pi}_\theta(\mathbf{x}) \equiv \tilde{D}_0 \tilde{\Theta}|_{x^0=0}, \\ \tilde{\mathcal{A}}^k(\mathbf{x}) \equiv \tilde{\mathcal{A}}^k|_{x^0=0}, & \tilde{\mathcal{E}}_k(\mathbf{x}) \equiv \tilde{\mathcal{F}}^{0k}|_{x^0=0} \end{array} \tag{5.33}$$

are gauge-equivalent to $u_g X$:

$$\tilde{X} \sim u_g X. \tag{5.34}$$

System (5.28) for functions (5.32) is satisfied due to Poincare invariance of (5.28)l property (5.34) is checked by direct calculations for partial cases: spatial translations and rotations, evolution and boosts.

3. A tangent vector $\delta X \in T\mathcal{X}$ can be identified with a set $\delta\overline{X} \equiv (\delta\overline{S}, \delta\overline{\Phi}(x)) \in T\overline{\mathcal{X}}$; $\delta\overline{\Phi}$ being a solution to the variation system. Analogously to (2.23),

$$\begin{array}{c} \delta\{\partial_\nu \overline{F}^{\mu\nu} - i(\overline{\Theta}^* \overline{D}^\mu \overline{\Theta} - \overline{\Theta D^\mu \Theta^*})\} = 0; \\ \delta\{\overline{D}_\mu \overline{D}^\mu \overline{\Theta} + m^2 \overline{\Theta} + V'(\overline{\Theta}^* \overline{\Theta}) \overline{\Theta}\} = 0. \end{array} \tag{5.35}$$

Then one introduces the operator $\Omega[\delta\overline{\Phi}] = \Omega[\delta X]$. Notice that correspondence $(\delta\Pi, \delta\Phi) \mapsto \delta\overline{\Phi}$ is not one-to-one; however, one has $\Omega[\delta X] = 0$ if $X + \delta X \sim X$; therefore, the operator $\Omega[\delta\overline{\Phi}]$ is well-defined.

The commutation relation (2.15) between operators $\Omega[\delta\overline{\Phi}]$ can be taken to a manifestly covariant form. Making use of eq.(4.13), one obtains

$$\begin{gathered}[\Omega[\delta_1\overline{\Phi}],\Omega[\delta_2\overline{\Phi}]] = \\ -i\int_{x^0=0} d\mathbf{x}[\delta_1\mathcal{E}_k\delta_2\mathcal{A}^k + \delta_1\Pi_\theta\delta_2\Phi^* + \delta_1\Pi^*_\theta\delta_2\Phi - \delta_2\mathcal{E}_k\delta_1\mathcal{A}^k - \delta_2\Pi_\theta\delta_1\Phi^* - \delta_2\Pi^*_\theta\delta_1\Phi] = \\ -i\int_{x^0=0} d\mathbf{x}\left[\delta_1\frac{\partial\mathcal{L}}{\partial\overline{\Phi}_{,0}}\delta_2\overline{\Phi} - \delta_2\frac{\partial\mathcal{L}}{\partial\overline{\Phi}_{,0}}\delta_1\overline{\Phi}\right]\end{gathered}$$

with $\overline{\Phi} \equiv (\overline{\mathcal{A}}^\mu, \overline{\Theta}, \overline{\Theta}^*)$, the notation $\overline{\Phi}_{,\mu} \equiv \partial_\mu\overline{\Phi}$ is introduced, $\mathcal{L}$ is the classical Lagrangian

$$\mathcal{L} = \overline{D}_\mu\overline{\Theta}^*\overline{D}^\mu\overline{\Theta} - m^2\overline{\Theta}^*\overline{\Theta} - V(\overline{\Theta}^*\overline{\Theta}) - \frac{1}{4}\overline{\mathcal{F}}_{\mu\nu}\overline{\mathcal{F}}^{\mu\nu}.$$

One can notice that

$$\partial_\mu\left[\delta_1\frac{\partial\mathcal{L}}{\partial\overline{\Phi}_{,\mu}}\delta_2\overline{\Phi} - \delta_2\frac{\partial\mathcal{L}}{\partial\overline{\Phi}_{,\mu}}\delta_1\overline{\Phi}\right] = 0.$$

Therefore, the commutator relation is taken to the manifestly covariant form

$$[\Omega[\delta_1\overline{\Phi}],\Omega[\delta_2\overline{\Phi}]] = -i\int d\sigma_\mu\left[\delta_1\frac{\partial\mathcal{L}}{\partial\overline{\Phi}_{,\mu}}\delta_2\overline{\Phi} - \delta_2\frac{\partial\mathcal{L}}{\partial\overline{\Phi}_{,\mu}}\delta_1\overline{\Phi}\right] \tag{5.36}$$

4. Let us express the operator $\Xi O(X)$ via the operator 1-form Ω. Since $\Xi O(X)$ is a linear combination of semiclassical fields, it should be of the form

$$\Xi O(X) = -\Omega[\nabla_O\overline{\Phi}]. \tag{5.37}$$

Let us find an explicit form of $\nabla_O\overline{\Phi}$. It should be obtained from relation (2.40):

$$\delta O = \int d\sigma_\mu\left[\delta\frac{\partial\mathcal{L}}{\partial\overline{\Phi}_{,\mu}}\nabla_O\overline{\Phi} - \nabla_O\frac{\partial\mathcal{L}}{\partial\overline{\Phi}_{,\mu}}\delta\overline{\Phi}\right]. \tag{5.38}$$

One can construct variation $\nabla_O\overline{\Phi}$ in the following way. First, consider the function $\overline{\nabla_O\Phi}$ satisfying classical equations of motion with an external source:

$$\begin{gathered}\overline{\nabla_O}\left\{\frac{\partial\mathcal{L}}{\partial\overline{\Phi}} - \left(\frac{\partial\mathcal{L}}{\partial\overline{\Phi}_{,\alpha}}\right)_{,\alpha}\right\} = \frac{\delta O}{\delta\overline{\Phi}}, \\ \overline{\nabla_O\Phi}|_{x^0\to-\infty} = 0.\end{gathered} \tag{5.39}$$

Then, let $\nabla_O\Phi$ be a solution of variation system with boundary condition at $+\infty$:

$$\begin{gathered}\nabla_O\left\{\frac{\partial\mathcal{L}}{\partial\overline{\Phi}} - \left(\frac{\partial\mathcal{L}}{\partial\overline{\Phi}_{,\alpha}}\right)_{,\alpha}\right\} = 0, \\ \nabla_O\overline{\Phi} \equiv \nabla_O\overline{\Phi}|_{x^0=+\infty}.\end{gathered} \tag{5.40}$$

If the variation $\frac{\delta O}{\delta\overline{\Phi}(x)}$ is a function with compact support (the observable O is local), the limits $x^0 \to \pm\infty$ mean $x > supp\frac{\delta O}{\delta\overline{\Phi}(x)}$ and $x < \frac{\delta O}{\delta\overline{\Phi}(x)}$.

Let us check eq.(5.38). One takes the right-hand side to the form:

$$\begin{gathered}\int_{x^0\to+\infty} d\sigma_\mu \left[\delta\frac{\partial\mathcal{L}}{\partial\overline{\Phi}_{,\mu}}\overline{\nabla_O\Phi} - \overline{\nabla_O}\frac{\partial\mathcal{L}}{\partial\overline{\Phi}_{,\mu}}\delta\overline{\Phi}\right] = \int dx\partial_\mu \left[\delta\frac{\partial\mathcal{L}}{\partial\overline{\Phi}_{,\mu}}\overline{\nabla_O\Phi} - \overline{\nabla_O}\frac{\partial\mathcal{L}}{\partial\overline{\Phi}_{,\mu}}\delta\overline{\Phi}\right] = \\ \int dx \left[\delta\left(\frac{\partial\mathcal{L}}{\partial\overline{\Phi}_{,\mu}}\right)_{,\mu}\overline{\nabla_O\Phi} + \delta\frac{\partial\mathcal{L}}{\partial\overline{\Phi}_{,\mu}}(\overline{\nabla_O\Phi})_{,\mu} - \overline{\nabla_O}\left(\frac{\partial\mathcal{L}}{\partial\overline{\Phi}_{,\mu}}\right)_{,\mu}\delta\overline{\Phi} - \overline{\nabla_O}\frac{\partial\mathcal{L}}{\partial\overline{\Phi}_{,\mu}}(\delta\overline{\Phi})_{,\mu}\right].\end{gathered} \tag{5.41}$$

Making use of property (5.39) and relation

$$\delta\left\{\frac{\partial\mathcal{L}}{\partial\overline{\Phi}} - \left(\frac{\partial\mathcal{L}}{\partial\overline{\Phi}_{,\alpha}}\right)_{,\alpha}\right\} = 0,$$

one takes the right-hand side of expression (5.41) to the form

$$\int dx\frac{\delta O}{\delta\overline{\Phi}(x)}\delta\overline{\Phi}(x).$$

This coincides with the left-hand side δO. Therefore, formula (5.38) is satisfied and $\Xi O(X)$ has the form (5.37) under conditions (5.39) and (5.40). This is in agreement with analogous relations for scalar field theory (section 2).

An explicit form of eqs.(5.39) for scalar electrodynamics is

$$\begin{gathered}\overline{\nabla_O}\{\partial_\nu\overline{F}^{\mu\nu} - i(\overline{\Theta}^*\overline{D}^\mu\overline{\Theta} - \overline{\Theta}\overline{D}^\mu\overline{\Theta}^*)\} = -\frac{\delta O}{\delta\overline{\mathcal{A}}_\mu(x)}; \\ \overline{\nabla_O}\{\overline{D}_\mu\overline{D}^\mu\overline{\Theta} + m^2\overline{\Theta} + V'(\overline{\Theta}^*\overline{\Theta})\overline{\Theta}\} = \frac{\delta O}{\delta\overline{\Theta}^*(x)}; \\ \overline{\nabla_O}\{\overline{D}_\mu\overline{D}^\mu\overline{\Theta}^* + m^2\overline{\Theta}^* + V'(\overline{\Theta}^*\overline{\Theta})\overline{\Theta}^*\} = \frac{\delta O}{\delta\overline{\Theta}(x)}; \\ \overline{\nabla_O\mathcal{A}}^\mu|_{x^0\to-\infty} = 0, \quad \overline{\nabla_O\Theta}|_{x^0\to-\infty} = 0, \quad \overline{\nabla_O\Theta}^*|_{x^0\to-\infty} = 0\end{gathered} \tag{5.42}$$

and

$$\begin{gathered}\nabla_O\overline{\mathcal{A}}^\mu|_{x^0\to+\infty} = \overline{\nabla_O\mathcal{A}}^\mu|_{x^0\to+\infty}, \\ \nabla_O\overline{\Theta}|_{x^0\to+\infty} = \overline{\nabla_O\Theta}|_{x^0\to+\infty}, \quad \nabla_O\overline{\Theta}^*|_{x^0\to+\infty} = \overline{\nabla_O\Theta}^*|_{x^0\to+\infty}.\end{gathered}$$

Notice that variation system (5.42) is solvable iff the gauge invariance condition (5.26) is satisfied.

5. One can now check that relations (2.36) and (2.41) for ΞO are indeed satisfied. Property (2.36) is a corollary of Poincare invariance of system (5.42) and analogous property for the operators Ω. Property (2.41) is a corollary of gauge invariance of the observable O and property (2.31) for the operators Ω.

5.6 Conservation of gauge equivalence relation

An important property of semiclassical Poincare transformations is that they should take gauge equivalent semiclassical states to gauge equivalent (eq.(2.29)). Let us investigate this property for the more general case of semiclassical transformation:

$$(X^0, f^0) \mapsto (X^\tau = u_O^\tau X^0, f^\tau = U_O^\tau (u_O^\tau X \leftarrow X) f^0). \qquad (5.43)$$

One should check whether

$$(X_1, f_1) \sim (X_2, f_2)$$

implies that

$$(u_O^\tau X_1, U_O^\tau (u_O^\tau X_1 \leftarrow X_1) f_1) \sim (u_O^\tau X_2, U_O^\tau (u_O^\tau X_2 \leftarrow X_2) f_2). \qquad (5.44)$$

Property (5.44) is not convenient for check. It is more suitable to consider gauge-invariant sections of the semiclassical bundle. To specify a section χ, one should assign a quantum state $\chi_X \in \mathcal{F}_X$ to each $X \in \mathcal{X}$. We say that section χ is gauge invariant iff

$$\chi_{X_2} = V(X_2 \leftarrow X_1) \chi_{X_1} \qquad (5.45)$$

for all $X_1 \sim X_2$.

It is remarkable that semiclassical states (X_1, f_1) and (X_2, f_2) are gauge equivalent iff for all gauge invariant sections χ the relation

$$(\chi_{X_1}, f_1) = (\chi_{X_2}, f_2) \qquad (5.46)$$

is satisfied. Eq.(5.46) is a convenient necessary and sufficient condition of equivalence of semiclassical states.

An automorphism (5.43) of semiclassical bundle may be viewed as transformation in the space of sections. Namely, consider the operator $\check{U}_O^\tau$ taking section χ to the following section

$$(\check{U}_O^\tau \chi)_X = U_O^\tau (X \leftarrow u_O^{-\tau} X) \chi_{u_O^{-\tau} X}. \qquad (5.47)$$

It happens that property (5.44) means that gauge invariant sections are taken to gauge invariant. This can be checked by using identity (5.46).

To justify that the property of gauge invariance of section χ is conserved under time evolution, one can notice that the section $\chi_O^\tau \equiv \check{U}_O^\tau \chi$ satisfies the equation

$$i \frac{d}{d\tau} \chi_O^\tau = \check{O} \chi_O^\tau$$

with

$$\check{O} = \frac{1}{2}\Xi^2 O(X) + O^{(1)}(X) - i\nabla_O.$$

Condition of gauge invariance of the section can be written as

$$\begin{gathered}\check{\Lambda}[\alpha]\chi^\tau_O = 0, \quad \text{Hamiltonian gauge;} \\ \check{\Lambda}[\alpha]\chi^\tau_O = 0, \text{ and } \check{\mathcal{E}}_0[\kappa]\chi^\tau_O = 0, \quad \text{Lorentz gauge.}\end{gathered} \tag{5.48}$$

Therefore, one should check that

$$[\check{O}, \check{\Lambda}[\alpha]]\chi^\tau_O = 0, \quad [\check{O}, \check{\mathcal{E}}_0[\kappa]]\chi^\tau_O = 0 \tag{5.49}$$

under conditions (5.48).

It is shown in ref. [19] that for classical observables A and B

$$\begin{gathered}[i\nabla_A - \tfrac{1}{2}\Xi^2 A - A^{(1)}, i\nabla_B - \tfrac{1}{2}\Xi^2 B - B^{(1)}] = \\ i(i\nabla_{\{A;B\}} - \tfrac{1}{2}\Xi^2\{A;B\} + \nabla_B A^{(1)} - \nabla_A B^{(1)}),\end{gathered} \tag{5.50}$$

provided that the Weyl quantization is used. Here $\{A;B\}$ is a Poisson bracket. Thus, for the case of gauge-invariant observables,

$$\{O; \Lambda[\alpha]\} = 0, \qquad \{O; \mathcal{E}_0[\kappa]\} = 0 \tag{5.51}$$

on the constraint surface, properties (5.49) are formally satisfied. However, one should be careful with quantum corrections $O^{(1)}$ due to divergences and renormalization.

For the Poincare generators, one has:

- for the Hamiltonian gauge,

$$\begin{gathered}\{\Lambda_{\mathbf{x}}; \mathcal{H}\} = 0, \qquad \{\Lambda_{\mathbf{x}}; \mathcal{M}^{k0}\} = 0, \\ \{\Lambda_{\mathbf{x}}; \mathcal{P}^l\} = \partial_l \Lambda_{\mathbf{x}}, \qquad \{\Lambda_{\mathbf{x}}; \mathcal{M}^{kl}\} = (x^k\partial_l - x^l\partial_k)\Lambda_{\mathbf{x}};\end{gathered}$$

- for the Lorentz gauge, $\{\Lambda_{\mathbf{x}}; \mathcal{P}^l\}$ and $\{\Lambda_{\mathbf{x}}; \mathcal{M}^{kl}\}$ are the same, while

$$\begin{gathered}\{\Lambda_{\mathbf{x}}; \mathcal{H}\} = \Delta_{\mathbf{x}}\mathcal{E}_0(\mathbf{x}); \qquad \{\lambda_{\mathbf{x}}; \mathcal{M}^{k0}\} = \partial_s x^k \partial_s \mathcal{E}_0(\mathbf{x}); \\ \{\mathcal{E}_0(\mathbf{x}); \mathcal{H}\} = \Lambda_{\mathbf{x}}; \qquad \{\mathcal{E}_0(\mathbf{x}); \mathcal{M}^{k0}\} = x^k \Lambda_{\mathbf{x}}; \\ \{\mathcal{E}_0(\mathbf{x}); \mathcal{P}^l\} = \partial_l \mathcal{E}_0(\mathbf{x}); \qquad \{\mathcal{E}_0(\mathbf{x}); \mathcal{M}^{kl}\} = (x^k\partial_l - x^l\partial_k)\mathcal{E}_0(\mathbf{x}).\end{gathered}$$

Relations (5.51) are satisfied.

5.7 On group and infinitesimal properties

The remaining property to be checked is eq.(2.32). It can be simplified. Consider the operator $\check{U}_g$ in the space of gauge invariant sectons χ:

$$(\check{U}_g \chi)_X = U_g(X \leftarrow u_{g^{-1}} X) \chi_{u_{g^{-1}} X}. \tag{5.52}$$

Relation (2.32) will be rewritten as

$$\check{U}_{g_1 g_2} \chi = \check{U}_{g_1} \check{U}_{g_2} \chi. \tag{5.53}$$

Therefore, the correspondence $g \mapsto \check{U}_g$ in the space of sections is a representation of the Poincare group. To check eq.(5.53), it is more convenient to justify infinitesimal analogs of (5.53):

$$\begin{gathered} \left[\check{P}^\lambda; \check{P}^\mu\right] \chi = 0; \qquad \left[\check{M}^{\lambda\mu}; \check{P}^\sigma\right] \chi = i(g^{\mu\sigma} \check{P}^\lambda - g^{\lambda\sigma} \check{P}^\mu) \chi; \\ \left[\check{M}^{\lambda\mu}; \check{M}^{\rho\sigma}\right] = -i(g^{\lambda\rho} \check{M}^{\mu\sigma} - g^{\lambda\sigma} \check{M}^{\mu\rho} + g^{\mu\sigma} \check{M}^{\lambda\rho} - g^{\mu\rho} \check{M}^{\lambda\sigma}) \chi \end{gathered} \tag{5.54}$$

under conditions (5.48). making use of relations (5.50), one reduces relation (5.54) to the classical formulas:

$$\begin{gathered} \{\mathcal{P}^\lambda; \mathcal{P}^\mu\} = 0; \qquad \{\mathcal{M}^{\lambda\mu}; \mathcal{P}^\sigma\} = i(g^{\mu\sigma} \mathcal{P}^\lambda - g^{\lambda\sigma} \mathcal{P}^\mu); \\ \{\mathcal{M}^{\lambda\mu}; \mathcal{M}^{\rho\sigma}\} = -i(g^{\lambda\rho} \mathcal{M}^{\mu\sigma} - g^{\lambda\sigma} \mathcal{M}^{\mu\rho} + g^{\mu\sigma} \mathcal{M}^{\lambda\rho} - g^{\mu\rho} \mathcal{M}^{\lambda\sigma}). \end{gathered} \tag{5.55}$$

For the Lorentz gauge, relations (5.55) are satisfied exactly, for the Hamiltonian gauge, they are valid on constraint surface. For the Couloumb gauge, one can reduce it to one of other gauges.

6 Conclusions

Thus, axioms of semiclassical scalar electrodynamics have been discussed. The considered approach is not manifestly covariant, so that a rigorous proof of properties of semiclassical theory (analog of [12]) is not easy.

It is possible to simplify the semiclassical theory. One should use a manifestly covariant semiclassical approach [27]. For this approach, axioms formulated here are also valid; however, it is BRST-BFV quantization that can be formulated in this way; on the other hand, the Hamiltonian approach of this paper is applicable to Hamiltonian and Couloumb gauges as well.

One can also investigate the semiclassical properties of the non-abelian gauge theories.

The author is going to clarify these problems in further publications.

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